Classical Electrodynamics

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Contents

Chapter 1

Introduction

In this chapter we give a brief introduction to electrodynamics and we relate it to current problems in modern physics.

1.1 The Cube of Physics

In order to orient ourselves in the space of physics theories let us consider what one might call the "cube of physics". Classical electrodynamics — the subject of this course — has its well-deserved place in the space of all theories. Theory space can be spanned by three axes labelled with the three most fundamental constants of Nature: Newton's gravitational constant

$$G = 6.6720 \times 10^{-11}\text{kg}^{-1}\text{m}^3\text{sec}^{-2}, \tag{1.1.1}$$

the velocity of light

$$c = 2.99792456 \times 10^8\text{m sec}^{-1}, \tag{1.1.2}$$

(which would deserve the name Einstein's constant), and Planck's quantum

$$h = 6.6205 \times 10^{-34}\text{kg m}^2\text{sec}^{-1}. \tag{1.1.3}$$

Actually, it is most convenient to label the three axes by G, $1/c$, and h. For a long time it was not known that light travels at a finite speed or that there is quantum mechanics. In particular, Newton's classical mechanics corresponds to $c = \infty \ \Rightarrow \ 1/c = 0$ and $h = 0$, i.e. it is non-relativistic and non-quantum, and it thus takes place along the G-axis. If we also ignore gravity and put

$G = 0$ we are at the origin of theory space doing Newton's classical mechanics but only with non-gravitational forces. Of course, Newton realized that in Nature $G \neq 0$, but he couldn't take into account $1/c \neq 0$ or $h \neq 0$. Maxwell and the other fathers of electrodynamics had c built into their theory as a fundamental constant, and Einstein realized that Newton's classical mechanics needs to be modified to his theory of special relativity in order to take into account that $1/c \neq 0$. This opened a new dimension in theory space which extends Newton's line of classical mechanics to the plane of relativistic theories. When we take the limit $c \to \infty \quad \Rightarrow \quad 1/c \to 0$ special relativity reduces to Newton's non-relativistic classical mechanics. The fact that special relativity replaced non-relativistic classical mechanics does not mean that Newton was wrong. Indeed, his theory emerges from Einstein's in the limit $c \to \infty$, i.e. if light would travel with infinite speed. As far as our everyday experience is concerned this is practically the case, and hence for these purposes Newton's theory is sufficient. There is no need to convince a mechanical engineer to use Einstein's theory because her airplanes are not designed to reach speeds anywhere close to c. Once special relativity was discovered, it became obvious that there must be a theory that takes into account $1/c \neq 0$ and $G \neq 0$ at the same time. After years of hard work, Einstein was able to construct this theory — general relativity — which is a relativistic theory of gravity. The G-$1/c$-plane contains classical (i.e. non-quantum) relativistic and non-relativistic physics. Classical electrodynamics — the subject of this course — fits together naturally with general relativity, but is here considered in the absence of gravity, i.e. we'll assume $G = 0$ but $1/c \neq 0$. Since we are not yet studying the quantum version of electrodynamics (QED) we also assume $h = 0$.

A third dimension in theory space was discovered by Planck who started quantum mechanics and introduced the fundamental action quantum h. When we put $h = 0$ quantum physics reduces to classical physics. Again, the existence of quantum mechanics does not mean that classical mechanics is wrong. It is, however, incomplete and should not be applied to the microscopic quantum world. In fact, classical mechanics is entirely contained within quantum mechanics as the limit $h \to 0$, just like it is the $c \to \infty$ limit of special relativity. Quantum mechanics was first constructed non-relativistically (i.e. by putting $1/c = 0$). When we allow $h \neq 0$ as well as $1/c \neq 0$ (but put $G = 0$) we are doing relativistic quantum physics. This is where the quantum version of classical electrodynamics — quantum electrodynamics (QED) — is located in theory space. Also the entire standard model of elementary particle physics which includes QED as well as its analog for the strong force — quantum chromodynamics (QCD) — is located there. Today we know that there must be a consistent physical theory that allows $G \neq 0$, $1/c \neq 0$, and $h \neq 0$ all at the same time. However, this theory of relativistic quantum gravity has not yet been found, although there are some promising attempts using string theory.

The study of classical electrodynamics will open the door to a vast variety of interesting phenomena. For example, it can explain the origin of light and why the sky is blue, and it is at the basis for countless technical applications. Besides that, understanding classical electrodynamics is a prerequisite for progressing towards the modern theories of physics such as QED and QCD. This should be motivation enough to try to understand electrodynamics in great detail.

1.2 The Strength of Electromagnetism

Until now physicists have discovered four fundamental forces: gravity, electromagnetism, as well as the weak and strong nuclear forces. The first fundamental force to be understood was gravity. Newton described it mathematically in the middle of the seventeenth century. In our everyday life we experience gravity as a rather strong force because the enormous mass M of the earth exerts an attractive gravitational force of magnitude

$$F = \frac{GMm}{R^2} \tag{1.2.1}$$

on any mass m (such as our body). Here G is a fundamental constant of Nature — Newton's gravitational constant — and R is earth's radius, i.e. the distance of the body of mass m at the earth's surface from the earth's center of gravity. The force of gravity acts universally on all massive objects. For example, two protons of mass M_p at a distance r exert a gravitational force

$$F_g = \frac{GM_p^2}{r^2} \tag{1.2.2}$$

on each other. However, protons carry an electric charge e and thus they also exert a repulsive electrostatic Coulomb force

$$F_e = \frac{e^2}{r^2} \tag{1.2.3}$$

on each other. Electromagnetic forces are less noticeable in our everyday life because protons are usually contained inside the nucleus of electrically neutral atoms, i.e. their charge e is screened by an equal and opposite charge $-e$ of an electron in the electron cloud of the atom. Still, as a force between elementary particles, electromagnetism is much stronger than gravity. The ratio of the electrostatic and gravitational forces between two protons is

$$\frac{F_e}{F_g} = \frac{e^2}{GM_p^2}. \tag{1.2.4}$$

The strength of electromagnetic interactions is determined by the fine-structure constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036}, \tag{1.2.5}$$

which is constructed from fundamental constants of Nature: the basic charge quantum e, the velocity of light c and Planck's quantum divided by 2π, $\hbar = h/2\pi$. The fine-structure constant is a dimensionless number and hence completely independent of arbitrary choices of physical units. Today we do not understand why it has the above experimentally determined value.

The strength of gravitational interactions is determined by Newton's constant

$$G = \frac{\hbar c}{M_{\text{Planck}}^2}, \tag{1.2.6}$$

and can be expressed in terms of the Planck mass M_{Planck} which is the highest energy scale relevant in elementary particle physics

$$M_{\text{Planck}} = 1.3014 \times 10^{19} M_p. \tag{1.2.7}$$

Remarkably, it is qualitatively understood why the proton mass M_p is a lot smaller than the Planck mass. This is a consequence of the property of asymptotic freedom of quantum chromodynamics (QCD) — the quantum field theory that describes the dynamics of quarks and gluons inside protons.

Using eq.(1.2.5), eq.(1.2.6), as well as eq.(1.2.7), eq.(1.2.4) can be cast in the form

$$\frac{F_e}{F_g} = \frac{e^2}{GM_p^2} = \alpha \frac{M_{\text{Planck}}^2}{M_p^2} = 1.236 \times 10^{36}. \tag{1.2.8}$$

Hence, as a fundamental force electromagnetism is very much stronger than gravity. It is no surprise that understanding and learning how to manipulate a force as strong as electromagnetism has changed human life in countless ways. Since electric equipment surrounds us almost everywhere, it is hard to imagine life without it. Electromagnetism was first understood by Coulomb, Faraday, Maxwell, and other physicists in the eightteenth and nineteenth century. This culminated in a triumph of physics: the theoretical construction of Maxwell's equations as a result of numerous experimental observations, as well as the prediction of new physical phenomena that were subsequently experimentally verified. All this has led to numerous applications that have changed our life forever. Understanding the basic features of electromagnetism is the subject of this course.

1.3 Field Theory

During the twentieth century two more fundamental forces have been understood: the weak and strong nuclear forces. The strong force is even stronger than electromagnetism, but is again screened and thus not felt directly on macroscopic scales. Manipulation of the strong force has again changed the world forever. It holds the promise of energy supply through nuclear fission or fusion, but — more than electromagnetism — it can be dangerous, especially in the form of atomic bombs. Understanding the underlying fundamental forces has culminated in another triumph of physics. All we know today about the fundamental forces between elementary particles is summarized in the so-called standard model. The quest for understanding the fundamental forces is still ongoing. In the near future the Large Hadron Collider (LHC) at CERN will allow us to probe fundamental physics at even higher energy scales, which may open the door to new elementary particles and new fundamental forces.

Unlike the weak and strong nuclear forces which play a role only at distances as short as 1 fm $= 10^{-15}$ m, gravity and electromagnetism manifest themselves at macroscopic scales. This implies that, while the weak and strong nuclear forces must be treated quantum mechanically, gravity and electromagnetism can already be investigated using classical (i.e. non-quantum) physics. The investigation of classical electrodynamics has led to the introduction of a new powerful fundamental concept — the concept of fields. It has been discovered experimentally that "empty" space is endowed with physical entities that do not manifest themselves as material particles. Such entities are e.g. the electric and magnetic field. Each point in space and time has two vectors $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ attached to it. The electric and magnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ represent physical degrees of freedom associated with each spatial point $\vec{x}$. The physical reality of these abstract fields has been verified in numerous experiments. We should hence think of "empty" space as a medium with nontrivial physical properties. Maxwell's equations take the form

$$
\begin{aligned}
&\vec{\nabla}\cdot\vec{E}(\vec{x},t) = 4\pi\rho(\vec{x},t),\\
&\vec{\nabla}\times\vec{E}(\vec{x},t) + \frac{1}{c}\partial_t\vec{B}(\vec{x},t) = 0,\\
&\vec{\nabla}\cdot\vec{B}(\vec{x},t) = 0,\\
&\vec{\nabla}\times\vec{B}(\vec{x},t) - \frac{1}{c}\partial_t\vec{E}(\vec{x},t) = \frac{4\pi}{c}\vec{j}(\vec{x},t).
\end{aligned}
\tag{1.3.1}
$$

They are four coupled partial differential equations that allow us to determine the electromagnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ once the electric charge and current densities $\rho(\vec{x}, t)$ and $\vec{j}(\vec{x}, t)$ have been specified and some appropriate boundary and initial conditions have been imposed.

At first glance, the four Maxwell equations may look complicated and perhaps somewhat confusing. Once we understand better what they mean, we'll appreciate their beauty and elegance. These four equations describe all electromagnetic phenomena at the classical level, ranging from Coulomb's $1/r^2$ law to the origin of light. They summarize all results ever obtained over centuries of intense experimentation with electromagnetism, and they are the basis of countless electronic devices that we use every day. Maxwell's equations are an enormous achievement of nineteenth century physics. Numerous discoveries of the twentieth century would have been completely impossible to make without that prior knowledge. This should be reason enough to try to understand Maxwell's equations and their physical meaning in all detail.

The charge and current densities are the sources of the electromagnetic fields $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$. On the other hand, electromagnetic fields exert a force

$$\vec{F}(t) = q[\vec{E}(\vec{r}(t), t) + \frac{\vec{v}(t)}{c} \times \vec{B}(\vec{r}(t), t)] \tag{1.3.2}$$

on a particle with charge q at position $\vec{r}(t)$ and with the velocity $\vec{v}(t) = d\vec{r}(t)/dt$. The electric field exerts a force already on static charges, while the magnetic field exerts a Lorentz force only on moving charges.

The field concept has turned out to be of central importance far beyond classical electrodynamics. Also general relativity — Einstein's relativistic generalization of Newton's theory of gravity — is a field theory, and the quantum mechanical standard model of elementary particles is a so-called quantum field theory. In fact, the quantum mechanical version of electrodynamics — quantum electrodynamics (QED) — is an integral part of the standard model. Classical (i.e. non-quantum) electrodynamics is a physics student's first encounter with the field concept. Applying this concept and studying the classical dynamics of the electromagnetic field will allow us to understand numerous phenomena ranging from the forces between charged particles to the origin of light. It will also put us in the position to understand the basis of countless technical applications of electricity and magnetism. And last but not least, it will be a first step towards reaching the frontier of knowledge in fundamental physics which is currently defined by the standard model of particle physics.

1.4 Electrodynamics and Relativity

Electrodynamics, the theory of electric and magnetic fields, is also a theory of light. Light waves, excitations of the electromagnetic field, travel with the velocity c. Remarkably, as was found experimentally by Michelson and Morley in 1895, light always travels with the speed c, independent of the speed of an observer. For example, if a space ship leaves the earth at a speed $c/2$ and shines light back to the earth, we see the light approaching us with the velocity c, not with $c/2$. This was correctly described by Einstein in his theory of special relativity in 1905. Nevertheless, as a theory of light, Maxwell's equations of 1865 had relativity built in from the start. It was soon realized, e.g. by Lorentz and Poincaré, that there is a fundamental conflict between electrodynamics and Newton's classical mechanics. Both theories assume different fundamental structures of space and time and are thus inconsistent. Only after Einstein generalized Newton's classical mechanics to his theory of special relativity, this conflict was resolved.

1.5 Unification of Forces

Electricity and magnetism had not immediately been identified as related phenomena. If you rub a cat's skin against some other material, charge is transferred from one to the other, followed by an electric discharge that again neutralizes the two materials. A compass needle is influenced by the earth's magnetic field and can be distracted by approaching it with a magnetized piece of iron. What does a cat's skin have to do with a compass needle or a piece of iron? It took an enormous amount of experimentation and theoretical abstraction to realize that the above effects both belong to the large class of electromagnetic phenomena. Thanks to the previous work of generations of ingenious physicists, today we know that electricity and magnetism are unified to electromagnetism. It is a challenge for current generations of researchers to push the boundaries of knowledge further into the unknown. Can electrodynamics be unified with the weak and strong nuclear forces, with gravity, or even with other yet undiscovered fundamental forces? Before we can hope to contribute to these exciting questions of current research, we must work hard to understand what Coulomb, Faraday, Maxwell, and many other physicists of the eighteenth and nineteenth century have taught us about classical electrodynamics. Nothing more and nothing less than this is the subject of this course.

Chapter 2

Electric Charge

Charge is an important property of matter, which determines the interaction with the electromagnetic field. At microscopic scales charge is carried by electrons and protons. At those scales the dynamics of charges is described by quantum physics, in particular, by quantum electrodynamics (QED). Here we are concerned with macroscopic scales where we can use classical electrodynamics. At macroscopic scales the charge of matter is described by a charge density.

2.1 Charges of Elementary Particles

The basic units of electric charge are carried by elementary particles such as protons and electrons. Protons carry the charge $Q_p = e$ and electrons carry the charge $Q_e = -e$. Indeed charge is quantized in integer units of e. The quantization of charge is related to the consistency of the standard model of particle physics as a quantum field theory. The electrostatic Coulomb force between an electron and a proton is attractive and has magnitude

$$F_e = \frac{e^2}{r^2}. \tag{2.1.1}$$

Charge is an additive quantity of dimension mass$^{1/2}\times$length$^{3/2}\times$time^{-1}. Neutral atoms consist of an atomic nucleus containing Z protons of charge $Q_p = e$ and N neutrons of charge $Q_n = 0$, as well as an electron cloud containing Z electrons of charge $Q_e = -e$. Hence, the total charge of an atom is

$$Q_A = ZQ_p + NQ_n + ZQ_e = Ze + 0 + Z(-e) = 0. \tag{2.1.2}$$

Ions are atoms from which electrons have been removed by ionization. If one electron has been removed, the electric charge of the ion is

$$Q_I = ZQ_p + NQ_n + (Z-1)Q_e = Ze + 0 + (Z-1)(-e) = e. \qquad (2.1.3)$$

Today we know that protons and neutrons are not truly elementary objects. Indeed they consist of quarks and gluons which are permanently confined together by the strong nuclear force. While gluons are electrically neutral, quarks are charged. The electric charge of an up-quark is $Q_u = \frac{2}{3}e$ and the electric charge of a down-quark is $Q_d = -\frac{1}{3}e$. Protons consist of two up-quarks and one down-quark and thus indeed have charge

$$Q_p = 2Q_u + Q_d = 2\frac{2}{3}e - \frac{1}{3}e = e, \qquad (2.1.4)$$

while neutrons contain one up-quark and two down-quarks which implies

$$Q_n = Q_u + 2Q_d = \frac{2}{3}e - 2\frac{1}{3}e = 0. \qquad (2.1.5)$$

Since quarks carry fractional electric charges it may seem that charge is not quantized in integer units of e. However, single quarks cannot exist in isolation. They are confined and always occur together in groups of three inside protons and neutrons. In particular, an isolated fractional electric charge has never been observed in any experiment. The forces that confine quarks and gluons inside protons and neutrons are not of electromagnetic origin. Instead the strong force is caused by another type of charge — the so-called color (or chromo) charge — which is not additive but neutralizes when three quarks come together. The dynamics of quarks and gluons is determined by another quantum field theory — quantum chromodynamics (QCD), whose mathematical structure is similar to QED. In particular, both QED and QCD (and even the whole standard model of particle physics as well as general relativity) are gauge theories. The simplest gauge theory is classical electrodynamics, the theory that we'll be concerned with in this course.

2.2 Charges of Macroscopic Amounts of Matter

The matter that surrounds us consists of protons, neutrons, and electrons (or equivalently of quarks, gluons, and electrons). Still, when we are confronted with a macroscopic amount of matter, such as a block of metal, we need not always worry about the elementary particles it consists of. For example, when we want to accelerate the block by applying a given force, all we care about is its total mass. Up to tiny relativistic corrections this mass is just the sum of the masses of all protons, neutrons, and electrons contained in the block. The block of metal may also carry a net charge because it may not contain exactly the same number of protons and electrons. Indeed, since the number of protons and electrons inside a macroscopic amount of matter is enormous (of the order of Avogadro's number, i.e. 10^{23}), the mismatch between the number of protons and electrons may also be very large. Although the net charge of the block of metal will still be an integer multiple of the elementary charge e, that integer will typically be an enormous number. Thus the quantization of charge, which is essential for individual elementary particles, is practically irrelevant for macroscopic amounts of matter. This has consequences for the mathematical description of the charge distribution at macroscopic scales. Instead of describing the detailed positions of countless individual elementary charges, it is much more appropriate to define a charge density $\rho(\vec{x}, t)$. For this purpose, we consider a small spatial volume dV centered at the point $\vec{x}$. This volume is very small on macroscopic scales, let us say 10^{-6} m in diameter, but still large on microscopic scales (the typical size of an atom is 10^{-10} m) so that it still contains a very large number of elementary charges. The charge density $\rho(\vec{x}, t)$ at the point $\vec{x}$ is simply given by the net charge contained in the volume dV at time t divided by dV. Charge density has the dimension mass$^{1/2}$ $\times$ length$^{-3/2}$ $\times$ time^{-1}.

Moving charges give rise to electric currents. At macroscopic scales electric currents are described by a current density $\vec{j}(\vec{x}, t)$. The current density is given by the net amount of charge flowing through a small surface dS (again of say 10^{-6} m in diameter) perpendicular to the direction of $\vec{j}(\vec{x}, t)$ per area dS and time. Current density hence has the dimension mass$^{1/2}$ $\times$ length$^{-1/2}$ $\times$ time^{-2}. While the dynamics of individual elementary charges takes place at microscopic scales and thus requires the use of quantum physics, the macroscopic charge and current densities $\rho(\vec{x}, t)$ and $\vec{j}(\vec{x}, t)$ follow the rules of classical (i.e. non-quantum) physics.

2.3 Charge Conservation

Besides energy, momentum, and angular momentum, electric charge is a conserved quantity. In particular, there is no process between elementary particles in which the total amount of electric charge changes. Still the charge can be transferred from one particle to another. For example, an isolated neutron is unstable and eventually decays into a proton, an electron, and an anti-neutrino. Just like neutrons, neutrinos and their anti-particles are electrically neutral (i.e. $Q_\nu = 0$). However, neutrinos are much lighter than neutrons and were long believed to be massless. Only recently, it was found that neutrinos have a tiny mass. When a neutron decays into proton, electron, and anti-neutrino, the electric charge is conserved because

$$Q_n = Q_p + Q_e + Q_\nu. \tag{2.3.1}$$

Neutron decay is a process of the weak interactions mediated by the heavy bosons W^+ and W^- that were discovered at CERN in 1983. The W-bosons also carry electric charge, $Q_{W^\pm} = \pm e$. At a truly elementary level, a down-quark in the neutron emits a W^--boson and turns into an up-quark, thus turning the neutron into a proton. Again electric charge is conserved because

$$Q_d = -\frac{1}{3}e = \frac{2}{3}e - e = Q_u + Q_{W^+}. \tag{2.3.2}$$

The W^--boson is an unstable particle which subsequently decays into an electron and an anti-neutrino, and once again electric charge is conserved in this decay because

$$Q_{W^-} = -e = Q_e + Q_\nu. \tag{2.3.3}$$

If a neutron in our macroscopic block of metal decays, the anti-neutrino will escape unnoticed, but the total charge still remains the same. Hence, at macroscopic scales we need not worry about what the elementary constituents are doing in detail. All that matters are the resulting charge and current densities.

Still, the conservation of electric charge at the microscopic level has macroscopic consequences. At macroscopic scales, charge conservation manifests itself in the continuity equation

$$\partial_t \rho(\vec{x}, t) + \vec{\nabla} \cdot \vec{j}(\vec{x}, t) = 0. \tag{2.3.4}$$

This equation implies that the charge density can change with time, only if a current is flowing. In order to see this, let us integrate the continuity equation over some volume V. Using Gauss' integration theorem one then obtains

$$\partial_t Q(t) = \int_V d^3x \; \partial_t \rho(\vec{x}, t) = -\int_V d^3x \; \vec{\nabla} \cdot \vec{j}(\vec{x}, t) = -\int_{\partial V} d^2\vec{f} \cdot \vec{j}(\vec{x}, t) = -J(t). \tag{2.3.5}$$

Hence, the amount of charge $Q(t)$ contained in the volume V changes with time only if a net current $J(t)$ is flowing through the boundary surface ∂V.

Chapter 3

Electrostatics

Electrostatics is the theory of time-independent electric fields. Only two of the four Maxwell equations are relevant for electrostatics: Gauss' law and the time-independent version of Faraday's law.

3.1 Gauss' and Faraday's Laws

Electric charges are the sources of the electric field. This is the content of Gauss' law, which (in its time-dependent form) is one of Maxwell's equations. Here we consider Gauss' law for static electric fields

$$\vec{\nabla} \cdot \vec{E}(\vec{x}) = 4\pi\rho(\vec{x}), \tag{3.1.1}$$

where $\rho(\vec{x})$ is the charge density (charge per volume). The prefactor 4π is purely conventional and refers to the Gaussian system. In the literature one often finds another prefactor $1/\varepsilon_0$ which arises in the so-called MKS system. In these lectures we'll use the Gaussian system. A conversion table from the Gaussian to the MKS system is contained in appendix A.

The other basic equation of electrostatics is the time-independent version of Faraday's law

$$\vec{\nabla} \times \vec{E}(\vec{x}) = 0. \tag{3.1.2}$$

Gauss' law is a single scalar equation, while Faraday's law is a vector equation which thus corresponds to a set of three equations. Together Gauss' and Faraday's law hence represent four equations for the three components of $\vec{E}(\vec{x})$. One might think that Faraday's equation alone would be enough to determine $\vec{E}(\vec{x})$. However, this equation alone only implies

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}), \tag{3.1.3}$$

where $\Phi(\vec{x})$ is the electrostatic scalar potential. The minus-sign is purely conventional. By inserting eq.(3.1.3) into eq.(3.1.2), one realizes that Faraday's law is indeed satisfied because

$$\vec{\nabla} \times \vec{\nabla}\Phi(\vec{x}) = 0 \tag{3.1.4}$$

for any arbitrary function $\Phi(\vec{x})$. Hence, Faraday's law alone does not determine $\vec{E}(\vec{x})$ completely. We still need to determine the scalar potential $\Phi(\vec{x})$ for which we need one single equation. Inserting eq.(3.1.3) into Gauss' law eq.(3.1.1) we obtain

$$\vec{\nabla} \cdot \vec{\nabla}\Phi(\vec{x}) = \Delta\Phi(\vec{x}) = -4\pi\rho(\vec{x}), \tag{3.1.5}$$

which is known as the Poisson equation. In electrostatics we assume a given charge density $\rho(\vec{x})$ and solve the Poisson equation for $\Phi(\vec{x})$ after imposing appropriate boundary conditions. Then we determine the electric field from eq.(3.1.3).

Gauss' and Faraday's law can also be expressed in integral form. First, let us integrate Gauss' law eq.(3.1.1) over some volume V bounded by the surface $S = \partial V$. Using Gauss' integration theorem one then obtains

$$\int_S d^2\vec{f} \cdot \vec{E}(\vec{x}) = \int_V d^3x \ \vec{\nabla} \cdot \vec{E}(\vec{x}) = 4\pi \int_V d^3x \ \rho(\vec{x}) = 4\pi Q(V). \tag{3.1.6}$$

In other words, the electric flux

$$\Phi_E(S) = \int_S d^2\vec{f} \cdot \vec{E}(\vec{x}) \tag{3.1.7}$$

through some closed surface S is determined by the total charge $Q(V)$ contained in the volume V, and the Gauss law in integral form thus reads

$$\Phi_E(S) = 4\pi Q(V). \tag{3.1.8}$$

Similarly, we can integrate Faraday's law eq.(3.1.2) over another surface S which is bounded by a closed curve $\mathcal{C}$, i.e. $\partial S = \mathcal{C}$. Using Stoke's integration theorem we then obtain

$$\int_{\mathcal{C}} d\vec{l} \cdot \vec{E}(\vec{x}) = \int_S d^2\vec{f} \cdot \vec{\nabla} \times \vec{E}(\vec{x}) = 0. \tag{3.1.9}$$

In other words, the electric circulation vanishes, i.e.

$$\Omega_E(\mathcal{C}) = \int_{\mathcal{C}} d\vec{l} \cdot \vec{E}(\vec{x}) = 0, \tag{3.1.10}$$

which is Faraday's law in integral form.

3.2 Coulomb's Law

Let us consider a spherically symmetric charge distribution

$$\rho(\vec{x}) = \rho(|\vec{x}|) = \rho(r), \tag{3.2.1}$$

with $\rho(r) = 0$ for $|\vec{x}| = r > R$. The total charge (contained inside a 3-dimensional ball B^3_R of radius R) is then given by

$$Q = \int_{B^3_R} d^3x \ \rho(\vec{x}). \tag{3.2.2}$$

What is the electric field generated by this charge distribution? For symmetry reasons we know that a spherical charge distribution must generate a radial electric field, i.e.

$$\vec{E}(\vec{x}) = E(r)\vec{e}_r. \tag{3.2.3}$$

Here $r = |\vec{x}|$ is the distance from the origin and $\vec{e}_r = \vec{x}/r$ is the radial unit vector. Let us first consider distances $r > R$, i.e. we are in the outer region which is free of charges. We can now use Gauss' mathematical integration theorem applied to the physical Gauss law, i.e.

$$4\pi Q = \int_{B^3_r} d^3x \ 4\pi\rho(\vec{x}) = \int_{B^3_r} d^3x \ \vec{\nabla}\cdot\vec{E}(\vec{x}) = \int_{S^2_r} d^2\vec{f}\cdot\vec{E}(\vec{x}) = 4\pi r^2 E(r). \tag{3.2.4}$$

Here $S^2_r = \partial B^3_r$ is the 2-dimensional sphere that forms the boundary of B^3_r, and $4\pi r^2$ is the area of S^2_r. We hence obtain

$$E(r) = \frac{Q}{r^2}. \tag{3.2.5}$$

The electric field of a spherical charge distribution in the exterior region (i.e. for $r > R$) is proportional to the total charge Q and it decays with the distance squared. We can also take the limit of a point charge by letting $R \rightarrow 0$.

A test charge q placed into the electric field at the point $\vec{r}$ will experience the static Coulomb force

$$\vec{F}(\vec{r}) = q\vec{E}(\vec{r}) = \frac{qQ}{r^2}\vec{e}_r. \tag{3.2.6}$$

The force is directed outward for like charges which repel each other and it is directed inward for opposite charges which attract each other.

Let us now consider the electric field in the interior of the ball B^3_R. Again, we use Gauss' integration theorem, but now we have $r < R$ such that

$$4\pi Q(r) = \int_{B^3_r} d^3x \ 4\pi\rho(\vec{x}) = \int_{B^3_r} d^3x \ \vec{\nabla}\cdot\vec{E}(\vec{x}) = \int_{S^2_r} d^2\vec{f}\cdot\vec{E}(\vec{x}) = 4\pi r^2 E(r). \tag{3.2.7}$$

Here $Q(r)$ is the charge contained in B_r^3 and now

$$E(r) = \frac{Q(r)}{r^2}. \tag{3.2.8}$$

Once $r > R$, all the charge Q is contained in B_r^3 and then $Q(r) = Q$. However, in general the field results only from the charge $Q(r)$ contained inside B_r^3.

We should also convince ourselves that not only Gauss' law but also Faraday's law is indeed satisfied. For this purpose we calculate

$$\vec{\nabla} \times \vec{E}(\vec{x}) = \vec{\nabla} \times [E(r)\vec{e}_r] = 0. \tag{3.2.9}$$

Here we have used the curl in spherical coordinates given in appendix C. Since indeed $\vec{\nabla} \times \vec{E}(\vec{x}) = 0$, we should be able to find a scalar potential $\Phi(\vec{x})$ such that $\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x})$. For the point charge Q we have

$$\Phi(\vec{x}) = \frac{Q}{r}, \tag{3.2.10}$$

which indeed implies

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}) = \frac{Q}{r^2}\vec{e}_r. \tag{3.2.11}$$

Here we have used the general form for the gradient in spherical coordinates, again from appendix C.

Finally, we want to convince ourselves that the Gauss law $\vec{\nabla}\cdot\vec{E}(\vec{x}) = 4\pi\rho(\vec{x})$ is indeed satisfied for our point charge Q. The general expression for the divergence of a vector field in spherical coordinates from appendix C yields

$$\vec{\nabla} \cdot \vec{E}(\vec{x}) = \frac{1}{r^2}\partial_r(r^2 E(r)) = \frac{1}{r^2}\partial_r(r^2\frac{Q}{r^2}) = 0, \tag{3.2.12}$$

which is the correct charge density for a point charge, except for the origin $\vec{x} = \vec{0}$. The charge density of a point charge located at the origin is indeed singular, and is not correctly captured by the above calculation. Alternatively, we can compute $\Delta\Phi(\vec{x}) = -4\pi\rho(\vec{x})$. The general expression for the Laplacian in spherical coordinates, once again given in appendix C, implies

$$\Delta\Phi(\vec{x}) = \frac{1}{r^2}\partial_r(r^2\partial_r\Phi(r)) = \frac{1}{r^2}\partial_r(r^2\partial_r\frac{Q}{r}) = 0. \tag{3.2.13}$$

Again, this calculation misses the singularity of the charge density of the point charge at the origin. In fact, this density is not a regular function but a so-called distribution — the Dirac δ-function. The mathematical expression that correctly describes the singularity at the origin is

$$\Delta\Phi(\vec{x}) = \Delta\frac{Q}{r} = -4\pi Q\delta(\vec{x}). \tag{3.2.14}$$

We will not yet fully define the δ-function at this point, but it obeys

$$\delta(\vec{x}) = 0, \text{ for } \vec{x} \neq \vec{0}, \tag{3.2.15}$$

as well as

$$\int_{B_r^3} d^3x\ \delta(\vec{x}) = 1, \text{ for any } \ r > 0. \tag{3.2.16}$$

3.3 Coulomb Field of a General Charge Distribution

The Maxwell equations are linear equations. This implies that the sum of two solutions of the equations is again a solution. This leads to a superposition principle for electromagnetic fields. Using the superposition principle, we will now construct the Coulomb field of a general charge distribution. Before we consider the general case, let us consider the field of two point charges q_1 and q_2 located at the positions $\vec{x}_1$ and $\vec{x}_2$, respectively. The Coulomb potential of each of the charges is given by

$$\Phi_i(\vec{x}) = \frac{q_i}{|\vec{x} - \vec{x}_i|}. \tag{3.3.1}$$

Due to the superposition principle, the total potential of both charges is given by

$$\Phi(\vec{x}) = \frac{q_1}{|\vec{x} - \vec{x}_1|} + \frac{q_2}{|\vec{x} - \vec{x}_2|}. \tag{3.3.2}$$

The corresponding electric field then takes the form

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}) = \frac{q_1}{|\vec{x} - \vec{x}_1|^2}\frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|} + \frac{q_2}{|\vec{x} - \vec{x}_2|^2}\frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|}. \tag{3.3.3}$$

Both the potential $\Phi(\vec{x})$ and the electric field $\vec{E}(\vec{x})$ are the sum of the potentials and electric fields of the individual charges.

Let us now consider a collection of N point charges q_i ($i \in \{1, ..., N\}$) located at positions $\vec{x}_i$. Then the potential is given by

$$\Phi(\vec{x}) = \sum_{i=1}^{N} \frac{q_i}{|\vec{x} - \vec{x}_i|}, \tag{3.3.4}$$

and the corresponding electric field is

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}) = \sum_{i=1}^{N} \frac{q_i}{|\vec{x}-\vec{x}_i|^2} \frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|}. \tag{3.3.5}$$

Since we now know the Coulomb field of any collection of point charges, it should be possible to also deduce the field of a general continuous charge distribution. For this purpose we imagine that space is divided into small cubes of volume d^3x centered at the points $\vec{x}_i$. Each cube contains a charge q_i. When the cubes are made smaller and smaller the ratio q_i/d^3x approaches the charge density $\rho(\vec{x}_i)$. We now consider the Coulomb potential of all charges in the limit $d^3x \to 0$ and replace a discrete sum by a continuous integral

$$\Phi(\vec{x}) = \sum_i \frac{q_i}{|\vec{x}-\vec{x}_i|} = \sum_i d^3x \frac{\rho(\vec{x}_i)}{|\vec{x}-\vec{x}_i|} \to \int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}. \tag{3.3.6}$$

Let us check if this potential indeed satisfies the Poisson equation

$$\Delta\Phi(\vec{x}) = -4\pi\rho(\vec{x}), \tag{3.3.7}$$

keeping in mind that

$$\Delta\frac{1}{|\vec{x}|} = \Delta\frac{1}{r} = -4\pi\delta(\vec{x}) \;\Rightarrow \Delta\frac{1}{|\vec{x}-\vec{x}\,'|} = -4\pi\delta(\vec{x}-\vec{x}\,'). \tag{3.3.8}$$

Hence, we obtain

$$\Delta\Phi(\vec{x}) = \int d^3x' \rho(\vec{x}\,')\Delta\frac{1}{|\vec{x}-\vec{x}\,'|} = -4\pi\int d^3x' \rho(\vec{x}\,')\delta(\vec{x}-\vec{x}\,'). \tag{3.3.9}$$

This gives the desired result only if

$$\int d^3x' \rho(\vec{x}\,')\delta(\vec{x}-\vec{x}\,') = \rho(\vec{x}). \tag{3.3.10}$$

Indeed, this equation is satisfied as a consequence of the fact that $\delta(\vec{x}-\vec{x}\,')$ is the Dirac δ-function. We will better understand the properties of this function later during this course. Let us also determine the electric field of the general charge distribution

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}) = -\int d^3x' \rho(\vec{x}\,')\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}\,'|} = \int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^2}\frac{\vec{x}-\vec{x}\,'}{|\vec{x}-\vec{x}\,'|}. \tag{3.3.11}$$

3.4 Potential Energy of a Test Charge

Let us move a test charge q through a general static electric field along some curve $\mathcal{C}$ starting from the point A and ending at the point B. When the position of the test charge is $\vec{r}(t)$, the electrostatic force on the test charge is given by

$$\vec{F}(t) = q\vec{E}(\vec{r}(t)). \tag{3.4.1}$$

Hence, the electric field is doing the work

$$W = q\int_{\mathcal{C}} d\vec{l}\cdot\vec{E}(\vec{r}) = -q\int_{\mathcal{C}} d\vec{l}\cdot\vec{\nabla}\Phi(\vec{r}) = -q[\Phi(B)-\Phi(A)], \tag{3.4.2}$$

and thus the product $q\Phi(\vec{r})$ measures the potential energy of the test charge. In particular, when a test charge is moved around a closed curve (i.e. when $A = B$) no net work is done. This also follows from Stoke's theorem. Let S be a surface bounded by the closed curve $\mathcal{C} = \partial S$. Then the work done by the field on the test charge is given by

$$W = q\int_{\mathcal{C}} d\vec{l}\cdot\vec{E}(\vec{r}) = q\int_{S} d^2\vec{f}\cdot\vec{\nabla}\times\vec{E}(\vec{r}) = 0. \tag{3.4.3}$$

The last equality is a direct consequence of Faraday's law $\vec{\nabla}\times\vec{E}(\vec{x}) = 0$.

3.5 Multipole Expansion

What is the field generated by a charge distribution $\rho(\vec{x})$ at large distances? This question has a meaningful answer when the charge is localized in some finite region of radius d, such that $\rho(\vec{x}) = 0$ for $r = |\vec{x}| > d$. The scalar potential of a general charge distribution is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}. \tag{3.5.1}$$

For $r = |\vec{x}| \gg d$ and for $|\vec{x}\,'| < d$, we can approximate

$$\frac{1}{|\vec{x}-\vec{x}\,'|} = \frac{1}{r} + ..., \tag{3.5.2}$$

such that at very large distances

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}\,')}{r} = \frac{Q}{r}. \tag{3.5.3}$$

Here

$$Q = \int d^3x' \ \rho(\vec{x}') \tag{3.5.4}$$

is just the total charge. Hence, at very large distances the field of any charge distribution looks like that of a point charge with the same total charge Q.

Let us also consider the corrections to this result. For this purpose we expand

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} + \frac{\vec{x} \cdot \vec{x}'}{r^3} + ..., \tag{3.5.5}$$

such that

$$\Phi(\vec{x}) = \frac{Q}{r} + \int d^3x' \frac{\vec{x} \cdot \vec{x}' \rho(\vec{x}')}{r^3} = \frac{Q}{r} + \frac{\vec{x} \cdot \vec{P}}{r^3}. \tag{3.5.6}$$

Here

$$\vec{P} = \int d^3x' \ \vec{x}' \rho(\vec{x}') \tag{3.5.7}$$

is the so-called electric dipole moment of the charge distribution. In analogy to the center of mass in classical mechanics, let us define a "center of charge" as

$$\vec{R} = \frac{\int d^3x' \ \rho(\vec{x}')\vec{x}'}{\int d^3x' \ \rho(\vec{x}')} = \frac{\vec{P}}{Q}. \tag{3.5.8}$$

In the "center of charge" system (the analog of the center of mass system) we have $\vec{R} = 0$. Hence, by an appropriate choice of the coordinate system one can then achieve $\vec{P} = 0$.

Let us also consider the case of a vanishing total charge $Q = 0$ such that

$$\Phi(\vec{x}) = \frac{\vec{x} \cdot \vec{P}}{r^3}. \tag{3.5.9}$$

For a charge distribution of total charge $Q = 0$, the electric dipole moment is independent of the choice of coordinate system. This follows when we shift the origin by a constant vector $\vec{a}$ such that

$$\vec{P}' = \int d^3x' \ (\vec{x}' + \vec{a})\rho(\vec{x}') = \vec{P} + \vec{a}Q = \vec{P}. \tag{3.5.10}$$

At large distances the electric field of a neutral electric dipole is given by

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}) = \frac{3(\vec{x} \cdot \vec{P})\vec{x} - r^2\vec{P}}{r^5}. \tag{3.5.11}$$

What is the next step in a systematic Taylor expansion? One obtains

$$\frac{1}{|\vec{x}-\vec{x}'|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + \frac{3(\vec{x}\cdot\vec{x}')^2 - r^2|\vec{x}'|^2}{2r^5} + ..., \tag{3.5.12}$$

such that now

$$\Phi(\vec{x}) = \frac{Q}{r} + \frac{\vec{x}\cdot\vec{P}}{r^3} + \sum_{i,j=1}^{3} \frac{3x_i x_j - \delta_{ij} r^2}{2r^5} Q_{ij}. \tag{3.5.13}$$

Here

$$Q_{ij} = \int d^3x' \ x'_i x'_j \rho(\vec{x}') \tag{3.5.14}$$

is a symmetric tensor (i.e. $Q_{ij} = Q_{ji}$) of so-called quadrupole moments. The trace of the tensor of quadrupole moments $\sum_{k=1}^{3} Q_{kk}$ determines the charge radius squared

$$a^2 = \frac{\int d^3x' \ |\vec{x}'|^2 \rho(\vec{x}')}{\int d^3x' \ \rho(\vec{x}')} = \frac{\sum_{k=1}^{3} Q_{kk}}{Q}. \tag{3.5.15}$$

Let us now construct a traceless tensor of quadrupole moments as

$$Q'_{ij} = Q_{ij} - \frac{1}{3}\sum_k Q_{kk}\delta_{ij}. \tag{3.5.16}$$

Obviously, the trace $\sum_{k=1}^{3} Q'_{kk} = 0$ indeed vanishes. The resulting field takes the form

$$\begin{aligned} \Phi(\vec{x}) &= \frac{Q}{r} + \frac{\vec{x}\cdot\vec{P}}{r^3} + \sum_{i,j=1}^{3} \frac{3x_i x_j - \delta_{ij} r^2}{2r^5} \left(Q'_{ij} + \frac{1}{3}\sum_{k=1}^{3} Q_{kk}\delta_{ij} \right) \\ &= \frac{Q}{r} + \frac{\vec{x}\cdot\vec{P}}{r^3} + \sum_{i,j=1}^{3} \frac{3x_i x_j - \delta_{ij} r^2}{2r^5} Q'_{ij}. \end{aligned} \tag{3.5.17}$$

The trace term proportional to $\sum_{k=1}^{3} Q_{kk}$ does not contribute to the field because

$$\sum_{i,j=1}^{3} \left(3x_i x_j - \delta_{ij} r^2\right) \delta_{ij} = 0. \tag{3.5.18}$$

The Taylor expansion can be carried out to higher orders and then leads to higher multipole moments. We will not consider the higher order terms here.

3.6 Electric Field at a Conducting Surface

So far we have discussed electrodynamics in the vacuum. Later we will extend our discussion to electromagnetism in media. As an intermediate step, we will now consider electrodynamics in a vacuum, but we will take into account the effect of matter via boundary conditions. This is possible if the matter acts as a perfect reflector for electromagnetic fields. Perfect mirrors do not really exist. Still, conductors are to a good approximation perfect reflectors. Here we make the idealization of a perfect conductor, i.e. we assume that charges in it move without resistance. A perfect conductor acts as a perfect mirror for electromagnetic fields. Mathematically, this is represented as a boundary condition at the surface of the conductor. Inside a perfect conductor the electric field vanishes because freely moving electrons inside the conductor would immediately eliminate any potential difference.

Let us consider the surface of a perfect conductor. The unit vector normal to the surface is denoted by $\vec{n}$. The component of the electric field in the direction of $\vec{n}$ (i.e. normal to the surface) is then given by $\vec{n} \cdot \vec{E}$ and the tangential component (along the surface) is related to $\vec{n} \times \vec{E}$. If the electric field had a non-zero tangential component, a surface current would immediately neutralize this component. Therefore

$$\vec{n} \times \vec{E}(\vec{x}) = 0 \qquad (3.6.1)$$

at the surface of a perfect conductor. A normal component of the electric field cannot be compensated in the same way because electrons cannot flow outside the conductor, and hence in general $\vec{n} \cdot \vec{E} \neq 0$. Let us integrate Gauss' law over the volume of a thin Gaussian pill box P at the surface of the conductor

$$\int_{\partial P} d^2 \vec{f} \cdot \vec{E}(\vec{x}) = \int_P d^3x \ \vec{\nabla} \cdot \vec{E}(\vec{x}) = 4\pi \int_P d^3x \ \rho(\vec{x}). \qquad (3.6.2)$$

Since the electric field vanishes inside the conductor the integral over the part of the boundary ∂P inside the conductor (the bottom of the pill box) vanishes. Since the transverse component of the electric field vanishes, the surface integral over the side of the pill box also vanishes. Finally, the surface integral over the top of the pill box outside the conductor gives $\vec{n} \cdot \vec{E}$ times the surface area of the cross-section of the box. The integral of the charge density over the volume P is the total charge inside the box. In the limit of a very thin box, only a surface charge density $\rho_s(\vec{x})$ contributes such that

$$\vec{n} \cdot \vec{E}(\vec{x}) = 4\pi\rho_s(\vec{x}) \qquad (3.6.3)$$

at the surface of a perfect conductor. The surface charge density $\rho_s(\vec{x})$ has the dimension of charge per area. The above equation is not a boundary condition for the electric field but just a formula for the surface charge density $\rho_s(\vec{x})$.

3.7 Point Charge near a Conducting Plate

Let us consider a conducting surface in the x-y-plane together with a point charge q at the position $\vec{x}_1 = (0, 0, L)$ on the z-axis. What is the Coulomb field of that charge and what is the induced surface charge? In particular, the ordinary Coulomb field of a point charge will now be modified due to the boundary condition for the electric field at the surface of the conductor. In our case, the boundary condition reads $E_x(x, y, 0) = E_y(x, y, 0) = 0$. While the solution of general problems of this kind requires the use of the so-called Green function method, our specific problem can be solved by a simple trick — the method of mirror image charges. The idea of this method is to mimic the effect of the conducting surface by a set of charges behind the conducting surface. In our case a single mirror charge $-q$ located at $\vec{x}_2 = (0, 0, -L)$ will do the job. The corresponding electric field is given by

$$\vec{E}(\vec{x}) = \frac{q}{|\vec{x} - \vec{x}_1|^2} \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|} - \frac{q}{|\vec{x} - \vec{x}_2|^2} \frac{\vec{x} - \vec{x}_2}{|\vec{x} - \vec{x}_2|}. \tag{3.7.1}$$

In particular

$$\begin{aligned} E_x(x, y, 0) &= \frac{qx}{\sqrt{x^2 + y^2 + L^2}^3} - \frac{qx}{\sqrt{x^2 + y^2 + L^2}^3} = 0, \\ E_y(x, y, 0) &= \frac{qy}{\sqrt{x^2 + y^2 + L^2}^3} - \frac{qy}{\sqrt{x^2 + y^2 + L^2}^3} = 0. \end{aligned} \tag{3.7.2}$$

In order to determine the induced surface charge density we calculate

$$\begin{aligned} 4\pi\rho_s(x, y) &= \vec{n} \cdot \vec{E}(x, y, 0) = E_z(x, y, 0) \\ &= -\frac{qL}{\sqrt{x^2 + y^2 + L^2}^3} - \frac{qL}{\sqrt{x^2 + y^2 + L^2}^3} \\ &= -\frac{2qL}{\sqrt{r^2 + L^2}^3}. \end{aligned} \tag{3.7.3}$$

Here we have introduced $r^2 = x^2 + y^2$. Let us also determine the total surface charge by integrating over the x-y-plane

$$\begin{aligned} Q_s &= \int dx\ dy\ \rho_s(x,y) = 2\pi \int_0^\infty dr\ r\rho_s(x,y) \\ &= -\int_0^\infty dr\ r\frac{qL}{\sqrt{r^2+L^2}^3} = \frac{qL}{\sqrt{r^2+L^2}}|_0^\infty = -q. \end{aligned} \tag{3.7.4}$$

Hence, the total surface charge is equal and opposite to the point charge q.

3.8 Charge near a Grounded Conducting Sphere

We will now apply the method of mirror charges in a slightly more complicated case, namely for a charge near a spherical conducting surface of radius R around the origin. Again, we place the charge q at $(0,0,L)$ on the z-axis, outside the conducting sphere (i.e. $L > R$). We want to ground the conducting sphere, i.e. we allow charges to flow on or off the sphere and we prescribe the constant potential $\Phi = 0$ at the surface. Interestingly, the effect of a grounded sphere can again be mimicked by a mirror image charge q' inside the sphere. For symmetry reasons, we again place the mirror charge on the z-axis at $(0,0,L')$. The total potential of both charges is then given by

$$\Phi(x,y,z) = \frac{q}{\sqrt{x^2+y^2+(z-L)^2}} + \frac{q'}{\sqrt{x^2+y^2+(z-L')^2}}. \tag{3.8.1}$$

Demanding $\Phi = 0$ at the surface of the sphere implies

$$\frac{q}{\sqrt{R^2-2zL+L^2}} = -\frac{q'}{\sqrt{R^2-2zL'+L'^2}}, \tag{3.8.2}$$

such that

$$q^2(R^2-2zL'+L'^2) = q'^2(R^2-2zL+L^2). \tag{3.8.3}$$

This can hold for all z only if

$$q^2(R^2+L'^2) = q'^2(R^2+L^2),\ q^2L' = q'^2L. \tag{3.8.4}$$

Solving these two equations for q' and L', we obtain

$$L' = \frac{R^2}{L} < R,\ q' = -q\frac{R}{L}, \tag{3.8.5}$$

such that

$$\Phi(x,y,z) = \frac{q}{\sqrt{x^2+y^2+(z-L)^2}} - \frac{qR}{L\sqrt{x^2+y^2+(z-R^2/L)^2}}. \tag{3.8.6}$$

Let us also determine the corresponding surface charge density. For this purpose, we first express the potential in spherical coordinates using $x^2+y^2+z^2 = r^2$ and $z = r\cos\theta$ such that

$$\Phi(r,\theta) = \frac{q}{\sqrt{r^2 + L^2 - 2Lr\cos\theta}} - \frac{qR}{L\sqrt{r^2 + R^4/L^2 - 2(R^2/L)r\cos\theta}}. \tag{3.8.7}$$

The surface charge density is given by the radial component of the electric field

$$\begin{aligned} E_r(r,\theta) &= -\partial_r\Phi(r,\theta) = \frac{q(r - L\cos\theta)}{\sqrt{r^2 + L^2 - 2Lr\cos\theta}^3} \\ &- \frac{qR(r - (R^2/L)\cos\theta)}{L\sqrt{r^2 + R^4/L^2 - 2(R^2/L)r\cos\theta}^3}. \end{aligned} \tag{3.8.8}$$

In particular, we get

$$4\pi\rho_s(\theta) = E_r(R,\theta) = \frac{q(R - L^2/R)}{\sqrt{R^2 + L^2 - 2LR\cos\theta}^3}, \tag{3.8.9}$$

which implies for the total induced surface charge

$$Q_s = 2\pi R^2 \int_0^\pi d\theta\ \sin\theta\ \rho_s(\theta) = -q\frac{R}{L} = q'. \tag{3.8.10}$$

Chapter 4

Magnetostatics

Magnetostatics is concerned with time-independent magnetic fields. The basic equations of magnetostatics are the magnetic Gauss law and the time-independent version of Ampere's law.

4.1 Magnetic Gauss Law and Ampere's Law

In contrast to electric charges, no magnetic charges — so-called magnetic monopoles — seem to exist in Nature. Indeed, every magnet that has ever been found had both a north and a south pole (i.e. there are only magnetic dipoles). The absence of magnetic charges is expressed mathematically in the magnetic Gauss law

$$\vec{\nabla} \cdot \vec{B}(\vec{x}) = 0, \tag{4.1.1}$$

whose time-dependent variant is one of Maxwell's equations. It can be solved by writing the magnetic field as the curl of a vector potential, i.e.

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x}). \tag{4.1.2}$$

Eq.(4.1.1) is then automatically satisfied because

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{A}(\vec{x}) = 0 \tag{4.1.3}$$

for any function $\vec{A}(\vec{x})$. The vector potential itself is determined via Ampere's law

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \frac{4\pi}{c} \vec{j}(\vec{x}). \tag{4.1.4}$$

Here $\vec{j}(\vec{x})$ is the current density — the amount of charge flowing through a surface element per time. Inserting eq.(4.1.2) into eq.(4.1.4) one obtains

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{A}(\vec{x})] = \vec{\nabla}[\vec{\nabla} \cdot \vec{A}(\vec{x})] - \Delta \vec{A}(\vec{x}) = \frac{4\pi}{c} \vec{j}(\vec{x}). \tag{4.1.5}$$

In magnetostatics one assumes that the current density $\vec{j}(\vec{x})$ is given and one then solves the above equation for the vector potential $\vec{A}(\vec{x})$ after imposing appropriate boundary conditions. Finally, the magnetic field $\vec{B}(\vec{x})$ is obtained from eq.(4.1.2).

Taking the divergence of eq.(4.1.4) we obtain

$$\vec{\nabla} \cdot \vec{j}(\vec{x}) = \frac{c}{4\pi} \vec{\nabla} \cdot \vec{\nabla} \times \vec{B}(\vec{x}) = 0, \tag{4.1.6}$$

which represents a constraint on $\vec{j}(\vec{x})$. Indeed this is just the continuity equation

$$\partial_t \rho(\vec{x}, t) + \vec{\nabla} \cdot \vec{j}(\vec{x}, t) = 0 \tag{4.1.7}$$

specialized to static fields.

Just like Gauss' and Faraday's law in electrostatics, the magnetic Gauss law and Ampere's law can also be expressed in integral form. Integrating the magnetic Gauss law eq.(4.1.1) over some volume V bounded by the surface $S = \partial V$, and using Gauss' integration theorem one obtains

$$\Phi_B(S) = \int_S d^2\vec{f} \cdot \vec{B}(\vec{x}) = \int_V d^3x \ \vec{\nabla} \cdot \vec{B}(\vec{x}) = 0. \tag{4.1.8}$$

In other words, the magnetic flux through any closed surface S vanishes, simply because there are no magnetic monopoles. Similarly, by integrating Ampere's law eq.(4.1.4) over another surface S bounded by a closed curve $\mathcal{C}$, i.e. $\partial S = \mathcal{C}$, and using Stoke's integration theorem we obtain

$$\int_{\mathcal{C}} d\vec{l} \cdot \vec{B}(\vec{x}) = \int_S d^2\vec{f} \cdot \vec{\nabla} \times \vec{B}(\vec{x}) = \frac{4\pi}{c} \int_S d^2\vec{f} \cdot \vec{j}(\vec{x}) = \frac{4\pi}{c} I(\mathcal{S}). \tag{4.1.9}$$

In other words, the magnetic circulation

$$\Omega_B(\mathcal{C}) = \int_{\mathcal{C}} d\vec{l} \cdot \vec{B}(\vec{x}) \tag{4.1.10}$$

along some closed curve $\mathcal{C}$ is determined by the total current $I(S)$ flowing through the surface S bounded by $\mathcal{C}$, i.e.

$$\Omega_B(\mathcal{C}) = \frac{4\pi}{c} I(\mathcal{S}), \tag{4.1.11}$$

which is Ampere's law in integral form.

4.2 Gauge Transformations

The magnetic Gauss law eq.(4.1.1) implies $\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})$. However, different vector potentials $\vec{A}(\vec{x})$ may give rise to the same magnetic field $\vec{B}(\vec{x})$. Indeed the transformed vector potential

$$\vec{A}(\vec{x})' = \vec{A}(\vec{x}) - \vec{\nabla}\varphi(\vec{x}) \tag{4.2.1}$$

(with $\varphi(\vec{x})$ being an arbitrary scalar function) yields the magnetic field

$$\vec{B}(\vec{x})' = \vec{\nabla} \times \vec{A}(\vec{x})' = \vec{\nabla} \times \vec{A}(\vec{x}) - \vec{\nabla} \times \vec{\nabla}\varphi(\vec{x}) = \vec{B}(\vec{x}), \tag{4.2.2}$$

since

$$\vec{\nabla} \times \vec{\nabla}\varphi(\vec{x}) = 0. \tag{4.2.3}$$

The transformation of eq.(4.2.1) is known as a gauge transformation. While the vector potential changes under the gauge transformation (i.e. it is gauge variant), the magnetic field remains the same (it is gauge invariant). The invariance under gauge transformations is an important symmetry of the laws of Nature, which is responsible for the conservation of the electric charge. Gauge symmetry indeed has turned out to be the key to all known fundamental forces including not only electromagnetism, but also the weak and strong nuclear forces as well as gravity. In particular, not only electrodynamics but the entire standard model of elementary particle physics is a gauge theory.

The gauge variance of the vector potential implies that we can make a gauge transformation such that $\vec{A}(\vec{x})'$ takes a particularly simple form. For example, one may decide to impose the so-called Coulomb gauge condition

$$\vec{\nabla} \cdot \vec{A}(\vec{x})' = 0, \tag{4.2.4}$$

which implies

$$\vec{\nabla} \cdot [\vec{A}(\vec{x}) - \vec{\nabla}\varphi(\vec{x})] = 0 \;\Rightarrow\; \Delta\varphi(\vec{x}) = \vec{\nabla} \cdot \vec{A}(\vec{x}). \tag{4.2.5}$$

Hence, fixing the Coulomb gauge requires to solve a Poisson equation for the gauge transformation function $\varphi(\vec{x})$. In the Coulomb gauge eq.(4.1.5) takes the simple form

$$\vec{\nabla} \times [\vec{\nabla} \times \vec{A}(\vec{x})'] = -\Delta\vec{A}(\vec{x})' = \frac{4\pi}{c}\vec{j}(\vec{x}). \tag{4.2.6}$$

4.3 Magnetic Field of a General Current Distribution

In electrostatics the general solution of the Poisson equation

$$\Delta\Phi(\vec{x}) = -4\pi\rho(\vec{x}) \tag{4.3.1}$$

is given by

$$\Phi(\vec{x}) = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}, \tag{4.3.2}$$

which implies

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}) = -\int d^3x' \rho(\vec{x}')\vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|} = \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|^2}\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|} \tag{4.3.3}$$

for the electric field of a general charge distribution. Similarly, in magnetostatics in the Coulomb gauge

$$\Delta\vec{A}(\vec{x})' = -\frac{4\pi}{c}\vec{j}(\vec{x}). \tag{4.3.4}$$

Hence, in analogy to electrostatics we can immediately write

$$\vec{A}(\vec{x})' = \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|}. \tag{4.3.5}$$

The magnetic field of a general current distribution is then given by

$$\vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x})' = \frac{1}{c}\int d^3x' \,\frac{\vec{j}(\vec{x}')\times(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}. \tag{4.3.6}$$

Here we have used

$$\vec{\nabla}\times\frac{\vec{j}(\vec{x}')}{|\vec{x}-\vec{x}'|} = \vec{\nabla}\frac{1}{|\vec{x}-\vec{x}'|}\times\vec{j}(\vec{x}') = -\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}\times\vec{j}(\vec{x}'). \tag{4.3.7}$$

It is a good exercise to check explicitly that the magnetic field of eq.(4.3.6) indeed satisfies the magnetic Gauss law as well as Ampere's law.

4.4 The Biot-Savart Law

The analog of Coulomb's law for a point charge in electrostatics is the Biot-Savart law of magnetostatics. The magnetic analog of a point charge in electrostatics is not a point magnetic charge, simply because magnetic monopoles do not exist in Nature. Instead the analog of a point charge is a current I flowing through a thin wire along some curve $\mathcal{C}$. The current density then has a fixed magnitude and its direction is tangential to the wire, i.e. it goes in the direction of the vector $d\vec{l}$ tangential to the curve $\mathcal{C}$. The corresponding magnetic field is thus given by the Biot-Savart law

$$\vec{B}(\vec{x}) = \frac{I}{c} \int_{\mathcal{C}} d\vec{l} \times \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3}. \tag{4.4.1}$$

Let us now consider the special case of a thin straight wire oriented along the z-direction. The symmetry of the problem suggests to use cylindrical coordinates. The magnetic field then takes the form $\vec{B}(\vec{x}) = B(\rho)\vec{e}_\varphi$ with

$$B(\rho) = \frac{I}{c} \int_{-\infty}^{\infty} dz \ \frac{\rho}{\sqrt{z^2 + \rho^2}^3} = \frac{2I}{c\rho}. \tag{4.4.2}$$

We can also confirm this result using symmetry considerations. In particular, we can apply Stokes' theorem by integrating Ampere's law

$$\vec{\nabla} \times \vec{B}(\vec{x}) = \frac{4\pi}{c} \vec{j}(\vec{x}) \tag{4.4.3}$$

over a circular disc S of radius ρ centered at the wire and perpendicular to it such that indeed

$$\frac{4\pi}{c} I = \frac{4\pi}{c} \int_S d^2\vec{f} \cdot \vec{j}(\vec{x}) = \int_S d^2\vec{f} \cdot \vec{\nabla} \times \vec{B}(\vec{x}) = \int_{\mathcal{C}} d\vec{l} \cdot \vec{B}(\vec{x}) = 2\pi\rho B(\rho). \tag{4.4.4}$$

4.5 Force between two Current Loops

Let us consider two current loops $\mathcal{C}_1$ and $\mathcal{C}_2$ carrying the currents I_1 and I_2. We want to compute the force that the two loops exert on each other. According to Newton's third law, the force that loop $\mathcal{C}_1$ exerts on loop $\mathcal{C}_2$ is equal and opposite to the force that $\mathcal{C}_2$ exerts on $\mathcal{C}_1$, which is the one we want to consider here. According to the Biot-Savart law, the current loop $\mathcal{C}_2$ generates a magnetic field

$$\vec{B}(\vec{x}_1) = \frac{I_2}{c} \int_{\mathcal{C}_2} d\vec{l}_2 \times \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3} \tag{4.5.1}$$

at a point $\vec{x}_1$ on the loop $\mathcal{C}_1$. The magnetic field gives rise to a Lorentz force

$$\vec{F}(\vec{x}_1) = q\frac{\vec{v}}{c} \times \vec{B}(\vec{x}_1) \tag{4.5.2}$$

on a charged particle moving with velocity $\vec{v}$ along the loop $\mathcal{C}_1$. Integrating the Lorentz force over the entire loop one obtains the total force

$$\vec{F} = \frac{I_1}{c}\int_{\mathcal{C}_1} d\vec{l}_1 \times \vec{B}(\vec{x}_1) = \frac{I_1 I_2}{c^2}\int_{\mathcal{C}_1}\int_{\mathcal{C}_2} d\vec{l}_1 \times \left[d\vec{l}_2 \times \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}\right]. \qquad (4.5.3)$$

Let us specialize the result to two parallel straight wires separated by a distance R carrying currents I_1 and I_2. The second wire runs along the z-axis, while the first wire is at $(R, 0, z)$. The force per length L between the two wires is then given by

$$\frac{\vec{F}}{L} = -\frac{I_1 I_2}{c^2} R \int_{-\infty}^{\infty} \frac{dz}{\sqrt{R^2 + z^2}^3}\vec{e}_x = -\frac{2 I_1 I_2}{c^2 R}\vec{e}_x. \qquad (4.5.4)$$

Somewhat counter-intuitively, the force is attractive if the currents I_1 and I_2 flow in the same direction and repulsive if they flow in opposite directions.

4.6 Magnetic Dipole Moment of a Current Distribution

In analogy to the electric multipole expansion let us consider a localized current distribution (which vanishes outside a finite region) and let us use

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} + \frac{\vec{x}\cdot\vec{x}'}{r^3} + ... \qquad (4.6.1)$$

To leading order the resulting vector potential in Coulomb gauge then takes the form

$$\vec{A}(\vec{x})' = \frac{1}{c}\int d^3x' \ \frac{\vec{j}(\vec{x}')}{|\vec{x} - \vec{x}'|} = \frac{1}{c}\int d^3x' \ \frac{\vec{j}(\vec{x}')}{r} = 0. \qquad (4.6.2)$$

In the last step, we have used the time-independent continuity equation $\vec{\nabla}\cdot\vec{j}(\vec{x}) = 0$ as well as

$$\int d^3x' \ j_i(\vec{x}') = \int d^3x' \ \sum_{k=1}^{3} \delta_{ik} j_k(\vec{x}') = \int d^3x' \ \sum_{k=1}^{3} \partial_k x'_i j_k(\vec{x}') =$$
$$-\int d^3x' \ x'_i \sum_{k=1}^{3} \partial_k j_k(\vec{x}') = -\int d^3x' \ x'_i \vec{\nabla}\cdot\vec{j}(\vec{x}') = 0. \qquad (4.6.3)$$

Here we have performed a partial integration. The next contribution is

$$\vec{A}(\vec{x})' = \frac{\vec{M}\times\vec{x}}{r^3}, \qquad (4.6.4)$$

with the magnetic dipole moment given by

$$\vec{M} = \frac{1}{2c} \int d^3x' \ \vec{x}' \times \vec{j}(\vec{x}'), \tag{4.6.5}$$

such that

$$\begin{aligned} \vec{A}(\vec{x})' &= \frac{1}{2c} \int d^3x' \ \frac{[\vec{x}' \times \vec{j}(\vec{x}')] \times \vec{x}}{r^3} \\ &= \frac{1}{2c} \int d^3x' \ \frac{(\vec{x} \cdot \vec{x}')\vec{j}(\vec{x}') - (\vec{x} \cdot \vec{j}(\vec{x}'))\vec{x}'}{r^3} \\ &= \frac{1}{c} \int d^3x' \ \frac{(\vec{x} \cdot \vec{x}')\vec{j}(\vec{x}')}{r^3}. \end{aligned} \tag{4.6.6}$$

In the last step we have used

$$\int d^3x' \ [j_i(\vec{x}')x'_k + j_k(\vec{x}')x'_i] = 0, \tag{4.6.7}$$

such that

$$\begin{aligned} \int d^3x' \ (\vec{x} \cdot \vec{j}(\vec{x}'))x'_i &= \int d^3x' \ \sum_{k=1}^{3} x_k j_k(\vec{x}')x'_i = -\int d^3x' \ \sum_{k=1}^{3} x_k j_i(\vec{x}')x'_k \\ &= -\int d^3x' \ (\vec{x} \cdot \vec{x}')j_i(\vec{x}'). \end{aligned} \tag{4.6.8}$$

Eq.(4.6.7) follows from

$$\begin{aligned} \int d^3x' \ j_i(\vec{x}')x'_k &= \int d^3x' \ \sum_{l=1}^{3} \delta_{il} j_l(\vec{x}')x'_k = \int d^3x' \ \sum_{l=1}^{3} \partial_l x'_i j_l(\vec{x}')x'_k \\ &= -\int d^3x' \ \sum_{l=1}^{3} x'_i j_l(\vec{x}')\partial_l x'_k = -\int d^3x' \ \sum_{l=1}^{3} x'_i j_l(\vec{x}')\delta_{kl} \\ &= -\int d^3x' \ j_k(\vec{x}')x'_i. \end{aligned} \tag{4.6.9}$$

The field of a magnetic dipole takes the form

$$\vec{B}(\vec{x}) = \vec{\nabla} \times \vec{A}(\vec{x})' = \frac{3(\vec{x} \cdot \vec{M}) \ \vec{x} - r^2\vec{M}}{r^5}, \tag{4.6.10}$$

and thus resembles the field of an electric dipole in electrostatics.

4.7 Magnetic Field at a Conducting Surface

In analogy to the discussion of boundary conditions for the electric field in electrostatics, let us now consider the boundary conditions for the magnetic field at the surface of a perfect conductor. By integrating Gauss' law over a thin Gaussian pill box at the surface of the conductor, we have shown that $\vec{n} \cdot \vec{E}(\vec{x}) = 4\pi\rho_s(\vec{x})$, where $\rho_s(\vec{x})$ is the electric surface charge density. Now we apply exactly the same argument to the magnetic Gauss law. Since there are no magnetic monopoles ($\vec{\nabla} \cdot \vec{B}(\vec{x}) = 0$), there are also no magnetic surface charges, and hence at the surface of a perfect conductor

$$\vec{n} \cdot \vec{B}(\vec{x}) = 0. \tag{4.7.1}$$

However, unlike for the electric field, in general $\vec{\nabla} \times \vec{B}(\vec{x}) \neq 0$. Let us now apply Stoke's theorem to integrate Ampere's law over a small rectangular closed curve $\mathcal{C}$ at the conductor's surface. Just as the electric field, the magnetic field inside a conductor vanishes. Thus, the circulation of the magnetic field around the closed curve gets a contribution only from the tangential component $\vec{n} \times \vec{B}(\vec{x})$ at the surface of the conductor. Using Stokes' theorem this equals the total current flowing through the loop which necessarily resides on the surface. Hence, at the surface of a perfect conductor we obtain

$$\vec{n} \times \vec{B}(\vec{x}) = \frac{4\pi}{c}\vec{j}_s(\vec{x}). \tag{4.7.2}$$

In analogy to electrostatics this equation is not a boundary condition for the magnetic field but just a formula that allows us to determine the surface current density $j_s(\vec{x})$.

4.8 Wire Parallel to a Conducting Plate

Let us consider a conducting plate in the x-y-plane with a straight wire along the y-direction carrying a current I parallel to the plane at a distance L. Just as the electric field of a point charge near a conducting plate was computed using a mirror charge, in this case we can use a mirror current to compute the magnetic field. While the original current flows at $(0, y, L)$, the mirror current flows at $(0, y, -L)$ in the opposite direction. Using the result for the magnetic field of a single straight wire along the z-axis

$$\vec{B}(\vec{x}) = \frac{2I}{c\rho}\vec{e}_\varphi \tag{4.8.1}$$

one obtains

$$B_x(\vec{x}) = -\frac{2I}{c}\left(\frac{L-z}{x^2+(z-L)^2} + \frac{L+z}{x^2+(z+L)^2}\right),$$
$$B_z(\vec{x}) = \frac{2I}{c}\left(\frac{x}{x^2+(z-L)^2} - \frac{x}{x^2+(z+L)^2}\right). \qquad (4.8.2)$$

As it should be, the perpendicular component $B_z(x, y, 0)$ vanishes at the surface $z = 0$ of the conductor. The induced surface current (flowing in the y-direction) is given by

$$\frac{4\pi}{c} j_s(x, y, 0) = B_x(x, y, 0) = -\frac{4I}{c}\frac{L}{x^2+L^2}. \qquad (4.8.3)$$

Integrating the induced surface current along the x-axis one obtains

$$\int_{-\infty}^{\infty} dx\ j_s(x, y, 0) = -\frac{I}{\pi}\int_{-\infty}^{\infty} dx\ \frac{L}{x^2+L^2} = -I. \qquad (4.8.4)$$

The total induced surface current is equal and opposite to the current flowing through the wire.

Chapter 5

Structure of Maxwell's Equations

Maxwell's equations are among the most beautiful and fundamental achievements of theoretical physics. They describe the classical (i.e. non-quantum) dynamics of the electromagnetic field. As such they are as fundamental as Newton's laws. Instead of following a historical route and inferring the various equations from different experimental observations, we'll simply take the equations as given, and derive everything else from there. In this chapter we'll explore some of the intriguing mathematical structures of Maxwell's equations, in particular their gauge structure and conserved physical quantities.

5.1 The Four Maxwell Equations

Charges are the sources of the electric field. The content of this sentence has a simple mathematical form

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = 4\pi\rho(\vec{x}, t), \tag{5.1.1}$$

which is one of Maxwell's four equations, also known as Gauss' law. Here $\vec{E}(\vec{x}, t)$ is the electric vector field at the space point $\vec{x}$ at time t and the scalar field $\rho(\vec{x}, t)$ is the electric charge density (charge per volume). As we have seen, Coulomb's $1/r^2$ force law is an immediate consequence of this equation.

There are no magnetic charges (so-called magnetic monopoles), i.e. there are no sources of the magnetic field, or in mathematical terms

$$\vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0, \tag{5.1.2}$$

which is another Maxwell equation, also known as the magnetic Gauss law. Here $\vec{B}(\vec{x}, t)$ is the magnetic field. The possible existence of magnetic monopoles was contemplated by the famous quantum physicist Dirac in 1931. However, despite numerous experimental searches for magnetically charged particles, not a single one has ever been found, and thus Maxwell's equation from above has indeed passed all experimental tests.

A temporal variation of the magnetic field implies a curl of the electric field, i.e.

$$\vec{\nabla} \times \vec{E}(\vec{x}, t) + \frac{1}{c}\partial_t \vec{B}(\vec{x}, t) = 0. \tag{5.1.3}$$

This Maxwell equation is also known as Faraday's law, which we'll investigate in more detail later.

Finally, the last of Maxwell's four equations relates the temporal variation of the electric field to a curl of the magnetic field and to the current density

$$\vec{\nabla} \times \vec{B}(\vec{x}, t) - \frac{1}{c}\partial_t \vec{E}(\vec{x}, t) = \frac{4\pi}{c}\vec{j}(\vec{x}, t). \tag{5.1.4}$$

Here $\vec{j}(\vec{x}, t)$ is the current density, which measures the amount of charge flowing through a surface element per time. The above equation is also referred to as Ampere's law. In the rest of this chapter we'll further explore the mathematical structure of the equations.

5.2 Maxwell's Equations in Integral Form

The Maxwell equations are a set of four coupled partial differential equations. Once the charge and current densities have been specified, and some appropriate boundary and initial conditions have been imposed, Maxwell's equations determine the spatial and temporal variations of the electromagnetic fields $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$. We will now cast Maxwell's equations in an integral form, using the theorems of Gauss and Stokes.

By integrating Gauss' law over a volume V and by using Gauss' integral theorem, one obtains the electric flux through the closed surface ∂V as

$$\Phi_E(t) = \int_{\partial V} d^2\vec{f} \cdot \vec{E}(\vec{x}, t) = \int_V d^3x \ \vec{\nabla} \cdot \vec{E}(\vec{x}, t) = 4\pi \int_V d^3x \ \rho(\vec{x}, t) = 4\pi Q(t). \tag{5.2.1}$$

Here $Q(t)$ is the amount of electric charge contained in the volume V at time t. Hence, by measuring the electric flux $\Phi_E(t)$ through a closed surface ∂V one can determine the total charge $Q(t)$ contained inside the volume V.

Similarly, by integrating the magnetic Gauss law over a volume V, one obtains an expression for the magnetic flux through any closed surface ∂V as

$$\Phi_B(t) = \int_{\partial V} d^2\vec{f} \cdot \vec{B}(\vec{x}, t) = \int_V d^3x \ \vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0. \tag{5.2.2}$$

Since there are no magnetic monopoles, the net magnetic flux through any closed surface vanishes.

The other two of Maxwell's equations can be cast in integral form by using Stokes' theorem. By integrating Faraday's law over some surface S one obtains

$$\begin{aligned} \Omega_E(t) &= \int_{\partial S} d\vec{l} \cdot \vec{E}(\vec{x}, t) = \int_S d^2\vec{f} \cdot \vec{\nabla} \times \vec{E}(\vec{x}, t) = -\frac{1}{c}\partial_t \int_S d^2\vec{f} \cdot \vec{B}(\vec{x}, t) \\ &= -\frac{1}{c}\partial_t \Phi_B(t). \end{aligned} \tag{5.2.3}$$

Hence, a change of the magnetic flux $\Phi_B(t)$ (in this case through a bounded surface) induces a circulation of the electric field around the closed curve ∂S.

By treating Ampere's law in a similar way we obtain

$$\begin{aligned} \Omega_B(t) &= \int_{\partial S} d\vec{l} \cdot \vec{B}(\vec{x}, t) = \int_S d^2\vec{f} \cdot \vec{\nabla} \times \vec{B}(\vec{x}, t) \\ &= \frac{1}{c}\partial_t \int_S d^2\vec{f} \cdot \vec{E}(\vec{x}, t) + \frac{4\pi}{c} \int_S d^2\vec{f} \cdot \vec{j}(\vec{x}, t) \\ &= \frac{1}{c}\partial_t \Phi_E(t) + \frac{4\pi}{c} I(t). \end{aligned} \tag{5.2.4}$$

Here $I(t)$ is the amount of charge flowing through the surface S per unit time and $\Phi_E(t)$ is the electric flux through that surface.

5.3 Charge Conservation

Electric charge is a conserved quantity, i.e. the total amount of charge in the Universe does not change with time. As far as we can tell, there are about as many positively as there are negatively charged particles and hence the total charge of the entire Universe seems to add up to zero. Charge conservation is as fundamental as the conservation of energy, momentum, or angular momentum and is a direct consequence of Maxwell's equations. In order to see this, let us consider the time-derivative of Gauss' law

$$\partial_t \rho(\vec{x}, t) = \frac{1}{4\pi} \partial_t \vec{\nabla} \cdot \vec{E}(\vec{x}, t), \tag{5.3.1}$$

and compare this with the divergence of Ampere's law

$$\vec{\nabla} \cdot \vec{j}(\vec{x}, t) = \frac{c}{4\pi} \vec{\nabla} \cdot \vec{\nabla} \times \vec{B}(\vec{x}, t) - \frac{1}{4\pi} \vec{\nabla} \cdot \partial_t \vec{E}(\vec{x}, t) = -\frac{1}{4\pi} \vec{\nabla} \cdot \partial_t \vec{E}(\vec{x}, t). \tag{5.3.2}$$

Here we have used $\vec{\nabla} \cdot \vec{\nabla} \times \vec{B}(\vec{x}, t) = 0$. Adding up eq.(5.3.1) and eq.(5.3.2) one obtains the continuity equation

$$\partial_t \rho(\vec{x}, t) + \vec{\nabla} \cdot \vec{j}(\vec{x}, t) = 0, \tag{5.3.3}$$

which indeed implies charge conservation.

5.4 Gauge Invariance

The electromagnetic fields $\vec{E}(\vec{x}, t)$ and $\vec{B}(\vec{x}, t)$ can be expressed in terms of scalar and vector potentials $\Phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ as

$$\begin{aligned} \vec{E}(\vec{x}, t) &= -\vec{\nabla} \Phi(\vec{x}, t) - \frac{1}{c} \partial_t \vec{A}(\vec{x}, t), \\ \vec{B}(\vec{x}, t) &= \vec{\nabla} \times \vec{A}(\vec{x}, t). \end{aligned} \tag{5.4.1}$$

Then the homogeneous Maxwell equations

$$\begin{aligned} &\vec{\nabla} \times \vec{E}(\vec{x}, t) + \frac{1}{c} \partial_t \vec{B}(\vec{x}, t) = \\ &-\vec{\nabla} \times \vec{\nabla} \cdot \Phi(\vec{x}, t) - \frac{1}{c} \vec{\nabla} \times \partial_t \vec{A}(\vec{x}, t) + \frac{1}{c} \partial_t \vec{\nabla} \times \vec{A}(\vec{x}, t) = 0, \\ &\vec{\nabla} \cdot \vec{B}(\vec{x}, t) = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A}(\vec{x}, t) = 0, \end{aligned} \tag{5.4.2}$$

are automatically satisfied. The inhomogeneous equations can be viewed as four equations for the four unknown functions $\Phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$.

All fundamental forces of Nature are described by gauge theories. This includes the electromagnetic, weak, and strong forces and even gravity. Gauge theories have a high degree of symmetry. In particular, their classical equations of motion (such as the Maxwell equations in the case of electrodynamics) are invariant against local space-time dependent gauge transformations. In electrodynamics a gauge transformation takes the form

$$\Phi(\vec{x},t)' = \Phi(\vec{x},t) + \frac{1}{c}\partial_t \varphi(\vec{x},t),$$
$$\vec{A}(\vec{x},t)' = \vec{A}(\vec{x},t) - \vec{\nabla}\varphi(\vec{x},t). \qquad (5.4.3)$$

Under this transformation the electromagnetic fields

$$\begin{aligned}\vec{E}(\vec{x},t)' &= -\vec{\nabla}\Phi(\vec{x},t)' - \frac{1}{c}\partial_t \vec{A}(\vec{x},t)' = -\vec{\nabla}\Phi(\vec{x},t) - \frac{1}{c}\partial_t \vec{A}(\vec{x},t) \\ &- \frac{1}{c}\vec{\nabla}\partial_t\varphi(\vec{x},t) + \frac{1}{c}\partial_t\vec{\nabla}\varphi(\vec{x},t) = \vec{E}(\vec{x},t), \\ \vec{B}(\vec{x},t)' &= \vec{\nabla}\times\vec{A}(\vec{x},t)' = \vec{\nabla}\times\vec{A}(\vec{x},t) - \vec{\nabla}\times\vec{\nabla}\varphi(\vec{x},t) = \vec{B}(\vec{x},t),\end{aligned} \qquad (5.4.4)$$

remain unchanged — they are gauge invariant. As a consequence, Maxwell's equations themselves are gauge invariant as well. In fact, in a gauge theory only gauge invariant quantities have a physical meaning. The scalar and vector potentials $\Phi(\vec{x},t)$ and $\vec{A}(\vec{x},t)$ vary under gauge transformations and are not physically observable. Instead they are mathematical objects with an inherent unphysical gauge ambiguity. Instead, as gauge invariant quantities, the electromagnetic fields $\vec{E}(\vec{x},t)$ and $\vec{B}(\vec{x},t)$ are physically observable.

5.5 Particle in an External Electromagnetic Field

The motion of a point particle is governed by Newton's equation

$$m\vec{a}(t) = \vec{F}(t). \qquad (5.5.1)$$

For a particle with charge q moving in an external electromagnetic field the force is given by

$$\vec{F}(t) = q[\vec{E}(\vec{r}(t),t) + \frac{\vec{v}(t)}{c}\times\vec{B}(\vec{r}(t),t)]. \qquad (5.5.2)$$

Newton's equation can be derived from the action

$$S[\vec{r}(t)] = \int dt\ \frac{m}{2}\vec{v}(t)^2 - \int dt d^3x\ [\rho(\vec{x},t)\Phi(\vec{x},t) - \vec{j}(\vec{x},t)\cdot\frac{1}{c}\vec{A}(\vec{x},t)], \qquad (5.5.3)$$

where

$$\rho(\vec{x},t) = q\delta(\vec{x}-\vec{r}(t)),$$
$$\vec{j}(\vec{x},t) = q\vec{v}(t)\delta(\vec{x}-\vec{r}(t)), \qquad (5.5.4)$$

are the charge and current densities of the charged particle at position $\vec{r}(t)$. It is easy to show that charge is conserved, i.e.

$$\partial_t \rho(\vec{x},t) + \vec{\nabla} \cdot \vec{j}(\vec{x},t) = 0. \tag{5.5.5}$$

Inserting eq.(5.5.4) into eq.(5.5.3), for the action one obtains

$$S[\vec{r}(t)] = \int dt \; [\frac{m}{2}\vec{v}(t)^2 - q\Phi(\vec{r}(t),t) + q\frac{\vec{v}(t)}{c} \cdot \vec{A}(\vec{r}(t),t)]. \tag{5.5.6}$$

This action is indeed invariant under gauge transformations because

$$\begin{aligned} &\int dt \; [\Phi(\vec{r}(t),t)' - \frac{\vec{v}(t)}{c} \cdot \vec{A}(\vec{r}(t),t)'] = \\ &\int dt \; [\Phi(\vec{r}(t),t) + \frac{1}{c}\partial_t \varphi(\vec{r}(t),t) - \frac{\vec{v}(t)}{c} \cdot (\vec{A}(\vec{r}(t),t) - \vec{\nabla}\varphi(\vec{r}(t),t))] = \\ &\int dt \; [\Phi(\vec{r}(t),t) - \frac{\vec{v}(t)}{c} \cdot \vec{A}(\vec{r}(t),t) + \frac{1}{c}\frac{d}{dt}\varphi(\vec{r}(t),t)], \end{aligned} \tag{5.5.7}$$

and because the total derivative

$$\frac{d}{dt}\varphi(\vec{r}(t),t) = \partial_t \varphi(\vec{r}(t),t) + \vec{v} \cdot \vec{\nabla}\varphi(\vec{r}(t),t), \tag{5.5.8}$$

integrates to zero as long as $\varphi(\vec{r}(t),t)$ vanishes in the infinite past and future. Identifying the Lagrange function

$$L = \frac{m}{2}\vec{v}(t)^2 - q\Phi(\vec{r}(t),t) + q\frac{\vec{v}(t)}{c} \cdot \vec{A}(\vec{r}(t),t), \tag{5.5.9}$$

it is straightforward to derive Newton's equation as the Euler-Lagrange equation

$$\frac{d}{dt}\frac{\delta L}{\delta v_i(t)} - \frac{\delta L}{\delta r_i} = 0. \tag{5.5.10}$$

The theory can also be formulated in terms of a classical Hamilton function

$$H = \vec{p}(t) \cdot \vec{v}(t) - L, \tag{5.5.11}$$

where $\vec{p}$ is the momentum canonically conjugate to the coordinate $\vec{r}$. One finds

$$m\vec{v}(t) = \vec{p}(t) - \frac{q}{c}\vec{A}(\vec{r}(t),t), \tag{5.5.12}$$

and thus one obtains

$$H = \frac{1}{2m}[\vec{p}(t) - \frac{q}{c}\vec{A}(\vec{r}(t),t)]^2 + q\Phi(\vec{r}(t),t). \tag{5.5.13}$$

This is indeed consistent because

$$v_i(t) = \frac{dr_i(t)}{dt} = \frac{\partial H}{\partial p_i(t)} = \frac{1}{m}[p_i(t) - \frac{q}{c}A_i(\vec{r}(t),t)]. \tag{5.5.14}$$

The other equation of motion is

$$\frac{dp_i(t)}{dt} = -\frac{\partial H}{\partial r_i(t)} = \frac{q}{mc}[p_j(t) - \frac{q}{c}A_j(\vec{r}(t),t)]\partial_i A_j(\vec{r}(t),t) + e\partial_i \Phi(\vec{r}(t),t). \tag{5.5.15}$$

It is straightforward to show that these equations of motion are again equivalent to Newton's equation.

5.6 Energy of the Electromagnetic Field

Let us consider a charged particle at position $\vec{r}(t)$ moving with velocity $\vec{v}(t) = d\vec{r}(t)/dt$ in an electromagnetic field. The particle then feels the force

$$\vec{F}(t) = q\left[\vec{E}(\vec{r}(t),t) + \frac{\vec{v}(t)}{c} \times \vec{B}(\vec{r}(t),t)\right]. \tag{5.6.1}$$

When the particle moves along some curve $\mathcal{C}$ the work done by the electromagnetic field is

$$W(t) = \int_{\mathcal{C}} d\vec{l} \cdot \vec{F}(t) = \int dt\ \frac{d\vec{r}(t)}{dt} \cdot \vec{F}(t) = \int dt\ \vec{v}(t) \cdot \vec{F}(t), \tag{5.6.2}$$

and hence the work done per unit of time is

$$\begin{aligned} \frac{dW(t)}{dt} &= \vec{v}(t) \cdot \vec{F}(t) = \vec{v}(t) \cdot q\left[\vec{E}(\vec{r}(t),t) + \frac{\vec{v}(t)}{c} \times \vec{B}(\vec{r}(t),t)\right] \\ &= q\vec{v}(t) \cdot \vec{E}(\vec{r}(t),t). \end{aligned} \tag{5.6.3}$$

Note that the magnetic field does not do work because $\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$. Only the electric field is doing work on the charged particle, namely

$$\frac{dW(t)}{dt} = \vec{I}(t) \cdot \vec{E}(\vec{r}(t),t), \tag{5.6.4}$$

where $\vec{I}(t) = q\vec{v}(t)$ is the electric current of the point particle. Let us now consider a continuous distribution of matter (not point particles) that has a current density $\vec{j}(\vec{x},t)$. Then the work done by the electromagnetic field per unit volume and per unit of time is given by

$$\frac{dw(\vec{x},t)}{dt} = \vec{j}(\vec{x},t) \cdot \vec{E}(\vec{x},t). \tag{5.6.5}$$

The work done by the electromagnetic field should give rise to an energy loss of the field itself. What is the energy of the electromagnetic field? To answer this question, let us consider the electric field in a capacitor

$$|\vec{E}| = 4\pi \frac{Q}{A}, \tag{5.6.6}$$

where A is the area of the capacitor plates and Q and $-Q$ are the total charges localized on the capacitor plates. If we now bring a small test charge dQ from one plate to the other, we must do the work

$$dW = dQ|\vec{E}|L = 4\pi L \frac{Q}{A} dQ, \tag{5.6.7}$$

where L is the distance between the plates. To charge the capacitor from zero to Q we hence need to do the total work

$$W = \frac{4\pi L}{A} \int_0^Q dQ\ Q = \frac{2\pi L}{A} Q^2. \tag{5.6.8}$$

This is the total energy stored in the capacitor (i.e. in the electric field between the plates). The energy density is thus given by

$$u = \frac{W}{AL} = 2\pi \frac{Q^2}{A^2} = \frac{1}{8\pi}|\vec{E}|^2. \tag{5.6.9}$$

It is proportional to the electric field squared. Similarly, energy is stored in the magnetic field. In general, the total energy density of the electromagnetic field is given by

$$u(\vec{x}, t) = \frac{1}{8\pi}(|\vec{E}(\vec{x}, t)|^2 + |\vec{B}(\vec{x}, t)|^2). \tag{5.6.10}$$

5.7 Energy Conservation and Poynting Vector

Let us compute how the energy density changes with time

$$\partial_t u(\vec{x}, t) = \frac{1}{4\pi} \left[\vec{E}(\vec{x}, t) \cdot \partial_t \vec{E}(\vec{x}, t) + \vec{B}(\vec{x}, t) \cdot \partial_t \vec{B}(\vec{x}, t)\right]. \tag{5.7.1}$$

Using the Maxwell equations

$$\vec{\nabla} \times \vec{E}(\vec{x}, t) + \frac{1}{c}\partial_t \vec{B}(\vec{x}, t) = 0,\ \vec{\nabla} \times \vec{B}(\vec{x}, t) - \frac{1}{c}\partial_t \vec{E}(\vec{x}, t) = \frac{4\pi}{c}\vec{j}(\vec{x}, t), \tag{5.7.2}$$

we obtain

$$\begin{aligned}
\partial_t u(\vec{x},t) &= \frac{1}{4\pi}[\vec{E}(\vec{x},t)\cdot(c\vec{\nabla}\times\vec{B}(\vec{x},t) - 4\pi\vec{j}(\vec{x},t)) \\
&- \vec{B}(\vec{x},t)\cdot c\vec{\nabla}\times\vec{E}(\vec{x},t)] \\
&= \frac{c}{4\pi}[\vec{E}(\vec{x},t)\cdot(\vec{\nabla}\times\vec{B}(\vec{x},t)) - \vec{B}(\vec{x},t)\cdot(\vec{\nabla}\times\vec{E}(\vec{x},t))] \\
&- \vec{j}(\vec{x},t)\cdot\vec{E}(\vec{x},t). \qquad (5.7.3)
\end{aligned}$$

We identify $\vec{j}\cdot\vec{E}$ as the work (per volume and time) done on the charged matter by the field. This decreases the energy of the electromagnetic field. What is the relevance of the other term? Vector analysis tells us

$$\vec{\nabla}\cdot\left[\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t)\right] = \vec{B}(\vec{x},t)\cdot(\vec{\nabla}\times\vec{E}(\vec{x},t)) - \vec{E}(\vec{x},t)\cdot(\vec{\nabla}\times\vec{B}(\vec{x},t)), \quad (5.7.4)$$

such that

$$\partial_t u(\vec{x},t) + \vec{\nabla}\cdot\vec{S}(\vec{x},t) + \vec{j}(\vec{x},t)\cdot\vec{E}(\vec{x},t) = 0. \quad (5.7.5)$$

Here we have introduced the quantity

$$\vec{S}(\vec{x},t) = \frac{c}{4\pi}\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t), \quad (5.7.6)$$

which is known as the Poynting vector. Eq.(5.7.5) expresses energy conservation and has the form of a continuity equation. This is particularly obvious in the absence of charged matter (i.e. for $\vec{j}=0$). The Poynting vector hence plays the role of an energy current. We can integrate eq.(5.7.5) over an arbitrary volume V with boundary ∂V and obtain

$$\begin{aligned}
\partial_t \int_V d^3x\ u(\vec{x},t) &= -\int_V d^3x\ \vec{\nabla}\cdot\vec{S}(\vec{x},t) - \int_V d^3x\ \vec{j}(\vec{x},t)\cdot\vec{E}(\vec{x},t) \\
&= -\int_{\partial V} d^2\vec{f}\cdot\vec{S}(\vec{x},t) - \int_V d^3x\ \vec{j}(\vec{x},t)\cdot\vec{E}(\vec{x},t). \quad (5.7.7)
\end{aligned}$$

The first term on the right hand side describes energy loss due to energy flowing out of the volume V through its surface ∂V.

Chapter 6

Electromagnetic Waves in Vacuum

Electromagnetic or light waves are excitations of the electromagnetic field. Here we consider the propagation of electromagnetic waves in the absence of charges and currents — i.e in the vacuum.

6.1 Plane Wave Solutions of the Wave Equations

Let us consider empty space, i.e. there is no matter and hence $\rho = 0$ and $\vec{j} = 0$. Maxwell's equations in vacuum then take the form

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} = 0, \ \vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0, \\ \vec{\nabla} \cdot \vec{B} = 0, \ \vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = 0. \end{aligned} \tag{6.1.1}$$

Hence, we obtain

$$\begin{aligned} \frac{1}{c^2}\partial_t^2 \vec{E} &= \frac{1}{c}\partial_t \vec{\nabla} \times \vec{B} = \vec{\nabla} \times \frac{1}{c}\partial_t \vec{B} \\ &= -\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = -\vec{\nabla}(\vec{\nabla} \cdot \vec{E}) + \Delta \vec{E} = \Delta \vec{E}, \\ \frac{1}{c^2}\partial_t^2 \vec{B} &= -\frac{1}{c}\partial_t \vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \frac{1}{c}\partial_t \vec{E} \\ &= -\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = -\vec{\nabla}(\vec{\nabla} \cdot \vec{B}) + \Delta \vec{B} = \Delta \vec{B}. \end{aligned} \tag{6.1.2}$$

These are wave equations for the electric and magnetic field, with wave propagation velocity c — the velocity of light.

Let us make the plane wave ansatz

$$\vec{E}(\vec{x},t) = \vec{f}(\vec{k}\cdot\vec{x} - \omega t) = \vec{f}(u),\ u = \vec{k}\cdot\vec{x} - \omega t. \tag{6.1.3}$$

Then

$$\partial_t^2 \vec{E} = \omega^2 \partial_u^2 \vec{f},\ \partial_x^2 \vec{E} = k_x^2 \partial_u^2 \vec{f},\ \partial_y^2 \vec{E} = k_y^2 \partial_u^2 \vec{f},\ \partial_z^2 \vec{E} = k_z^2 \partial_u^2 \vec{f}, \tag{6.1.4}$$

such that

$$\Delta\vec{E} = (\partial_x^2 + \partial_y^2 + \partial_z^2)\vec{E} = (k_x^2 + k_y^2 + k_z^2)\partial_u^2 \vec{f} = \frac{1}{c^2}\partial_t^2 \vec{E} = \frac{\omega^2}{c^2}\partial_u^2 \vec{f}, \tag{6.1.5}$$

which implies

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2 = |\vec{k}|^2 \ \Rightarrow\ \omega = c|\vec{k}|. \tag{6.1.6}$$

There is no dispersion because $d\omega/dk = c$, i.e. all electromagnetic waves travel with the same speed c. Still, the function $\vec{f}$ is restricted further because we must also satisfy $\vec{\nabla}\cdot\vec{E} = 0$. This implies

$$\vec{\nabla}\cdot\vec{E} = \partial_x E_x + \partial_y E_y + \partial_z E_z = k_x \partial_u f_x + k_y \partial_u f_y + k_z \partial_u f_z = \vec{k}\cdot\partial_u \vec{f} = 0. \tag{6.1.7}$$

Similarly, we write

$$\vec{B}(\vec{x},t) = \vec{g}(\vec{k}\cdot\vec{x} - \omega t) = \vec{g}(u),\ u = \vec{k}\cdot\vec{x} - \omega t, \tag{6.1.8}$$

such that $\vec{\nabla}\cdot\vec{B} = 0$ implies

$$\vec{k}\cdot\partial_u \vec{g} = 0. \tag{6.1.9}$$

We still must ensure that the other two Maxwell equations are also satisfied. For this purpose we consider

$$\begin{aligned} \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t \vec{B} &= \vec{k}\times\partial_u \vec{f} - \frac{\omega}{c}\partial_u \vec{g} = 0 \ \Rightarrow\ \partial_u \vec{g} = \frac{c}{\omega}\vec{k}\times\partial_u \vec{f}, \\ \vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t \vec{E} &= \vec{k}\times\partial_u \vec{g} + \frac{\omega}{c}\partial_u \vec{f} = 0. \end{aligned} \tag{6.1.10}$$

This is consistent because

$$\vec{k}\times\partial_u g + \frac{\omega}{c}\partial_u f = \vec{k}\times(\frac{c}{\omega}\vec{k}\times\partial_u \vec{f}) + \frac{\omega}{c}\partial_u \vec{f} = \frac{c}{\omega}(\vec{k}\cdot\partial_u \vec{f})\vec{k} - \frac{c}{\omega}k^2 \partial_u \vec{f} + \frac{\omega}{c}\partial_u \vec{f} = 0, \tag{6.1.11}$$

due to $\vec{k}\cdot\partial_u \vec{f} = 0$ and $k^2 = \omega^2/c^2$. All conditions we have derived can be satisfied simultaneously if $\vec{f}$ is perpendicular to $\vec{k}$, and $\vec{g}$ is perpendicular to both $\vec{k}$ and $\vec{f}$, i.e.

$$\vec{E}(\vec{x},t) = \vec{f}(\vec{k}\cdot\vec{x} - \omega t),\ \vec{B}(\vec{x},t) = \frac{c}{\omega}\vec{k}\times\vec{f}(\vec{k}\cdot\vec{x} - \omega t). \tag{6.1.12}$$

Still, $\vec{f}$ can be an arbitrary function as long as $\vec{k}\cdot\vec{f} = 0$.

6.2 The Energy of Electromagnetic Waves

For simplicity, let us consider an electromagnetic wave traveling in the z-direction with

$$\vec{E}(\vec{x},t) = E_0 \cos(kz - \omega t)\vec{e}_x, \ \vec{B}(\vec{x},t) = E_0 \cos(kz - \omega t)\vec{e}_y. \tag{6.2.1}$$

The energy density of the wave is given by

$$u = \frac{1}{8\pi}(|\vec{E}|^2 + |\vec{B}|^2) = \frac{1}{4\pi}E_0^2 \cos^2(kz - \omega t). \tag{6.2.2}$$

Now we average the energy density over one period $T = 2\pi/\omega$ and we obtain

$$\langle u \rangle = \frac{1}{T}\int_t^{t+T} dt \ \frac{1}{4\pi}E_0^2 \cos^2(kz - \omega t) = \frac{1}{8\pi}E_0^2. \tag{6.2.3}$$

We can also compute the Poynting vector

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} = \frac{c}{4\pi}E_0^2 \cos^2(kz - \omega t)\vec{e}_z. \tag{6.2.4}$$

Its temporal average is given by

$$\langle \vec{S} \rangle = \frac{c}{8\pi}E_0^2 \vec{e}_z \ \Rightarrow \ |\langle \vec{S} \rangle| = c\langle u \rangle. \tag{6.2.5}$$

This is characteristic of a traveling electromagnetic wave.

Let us also consider a standing wave

$$\vec{E}(\vec{x},t) = E_0 \sin(kz)\sin(\omega t)\vec{e}_x, \ \vec{B}(\vec{x},t) = E_0 \cos(kz)\cos(\omega t)\vec{e}_y. \tag{6.2.6}$$

In this case

$$u = \frac{1}{8\pi}E_0^2[\sin^2(kz)\sin^2(\omega t) + \cos^2(kz)\cos^2(\omega t)], \tag{6.2.7}$$

such that the temporal average is given by

$$\langle u \rangle = \frac{1}{8\pi}E_0^2[\frac{1}{2}\sin^2(kz) + \frac{1}{2}\cos^2(kz)] = \frac{1}{16\pi}E_0^2. \tag{6.2.8}$$

The Poynting vector is given by

$$\vec{S} = \frac{c}{4\pi}E_0^2 \sin(kz)\cos(kz)\sin(\omega t)\cos(\omega t)\vec{e}_z = \frac{c}{16\pi}E_0^2 \sin(2kz)\sin(2\omega t)\vec{e}_z, \tag{6.2.9}$$

such that $\langle \vec{S} \rangle = 0$. Hence, the time averaged Poynting flux vanishes and there is no net energy transport in a standing wave.

6.3 Radiation Pressure and Comet Tails

The electromagnetic field does not only carry energy, it also carries momentum and angular momentum. The momentum density of the field is given by

$$\vec{p}(\vec{x},t) = \frac{1}{4\pi c}\vec{E}(\vec{x},t) \times \vec{B}(\vec{x},t) = \frac{1}{c^2}\vec{S}(\vec{x},t). \tag{6.3.1}$$

This can be justified in analogy to the discussion of the energy. Returning to the traveling wave, where we found $|\langle\vec{S}\rangle| = c\langle u\rangle$, we now re-express this relation as

$$\langle u\rangle = \frac{1}{c}|\langle\vec{S}\rangle| = |\langle\vec{p}\rangle|c. \tag{6.3.2}$$

The energy density of the wave is given by the magnitude of its momentum density times the velocity of light.

In special special relativity, the energy of a free particle of rest mass M and momentum $\vec{P}$ is given by

$$E = \sqrt{(Mc^2)^2 + (|\vec{P}|c)^2}. \tag{6.3.3}$$

In particular, for a massless particle $E = |\vec{P}|c$. This is exactly what we found for the traveling wave. It behaves like a free massless particle. In fact, as a consequence of quantum mechanics, electromagnetic waves consist of massless particles called photons. However, $E = |\vec{P}|c$ and $\langle u\rangle = |\langle\vec{p}\rangle|c$ have nothing to do with quantum mechanics. These relations follow just from special relativity alone. This indicates that classical electrodynamics is indeed consistent with special relativity.

If a wave carries momentum, it should be possible to transfer this momentum to charged particles. This is indeed the case. Let us suppose that a wave is reflected by a mirror. Then the momentum of the wave changes its direction and a certain momentum is transferred to the mirror. If the wave hits the mirror at a 90^o angle, its momentum density $\vec{p}$ changes sign. During the time t the total momentum transferred to the mirror is

$$\vec{P} = 2\vec{p}Act, \tag{6.3.4}$$

where A is the area of the mirror. The force exerted on the mirror is hence given by

$$\vec{F} = \frac{d\vec{P}}{dt} = 2\vec{p}Ac = \frac{2}{c}\vec{S}A. \tag{6.3.5}$$

The magnitude of the force per unit area determines the pressure p (not to be confused with the momentum density $\vec{p}$) that the electromagnetic wave exerts on the mirror, i.e.

$$p = \frac{|\vec{F}|}{A} = \frac{2}{c}|\vec{S}|. \tag{6.3.6}$$

When the wave hits a perfect absorber instead of a perfect reflector, e.g. if it hits a perfectly black surface, the wave transfers all its momentum to the absorber. In this case the radiation pressure is only $p = |\vec{S}|/c$.

We know from experience that walking in the sun does not put us under any noticeable pressure. However, there are circumstances under which radiation pressure is not negligible. Lasers, for example, may generate very high radiation pressure. Physicists have even tried to ignite controlled nuclear fusion in this way. Even the radiation pressure of the sun may play a significant role, at least in outer space. Let us consider a particle of interstellar dust (perhaps from a comet's tail) in the gravitational field of the sun. Let us assume that the particle has radius r and mass density ρ and that it is a perfect absorber for sun light. The mass of the particle is then given by $m = 4\pi\rho r^3/3$ and is cross sectional area is $A = \pi r^2$. The particle will experience an attractive graviational force

$$F_g = \frac{GMm}{R^2} = \frac{4\pi}{3}\frac{GM\rho r^3}{R^2}, \tag{6.3.7}$$

where G is Newton's constant, R is the distance of the dust particle from the center of the sun, and M is the solar mass. What is the force that the sun exerts on the dust particle due to radiation pressure? We have

$$F_r = pA = \frac{|\vec{S}|}{c}\pi r^2. \tag{6.3.8}$$

The luminosity L of the sun is the total power that it emits, i.e.

$$L = 4\pi R^2|\vec{S}| \;\Rightarrow\; |\vec{S}| = \frac{L}{4\pi R^2}, \tag{6.3.9}$$

such that

$$F_r = \frac{Lr^2}{4cR^2}. \tag{6.3.10}$$

Both the gravitational force and the force due to radiation pressure fall off with $1/R^2$. Of course, while gravity is attractive, radiation pressure is repulsive. Under what circumstances do the two forces cancel each other?

$$F_g = F_r \;\Rightarrow\; \frac{4\pi}{3}\frac{GM\rho r^3}{R^2} = \frac{Lr^2}{4cR^2} \;\Rightarrow\; r = \frac{3L}{16\pi cMG\rho}. \tag{6.3.11}$$

Using the solar mass and luminosity as well as a reasonable number for ρ one estimates $r \approx 10^{-7}$m. Particles that are smaller than this critical value will be repelled from the sun. Radiation pressure explains why comet tails are always directed away from the sun.

Once we have appreciated that the electromagnetic field carries energy and momentum, it will come as no surprise that it also carries angular momentum. As always, angular momentum is defined with respect to some reference point, the origin of our coordinate system. The angular momentum density of the electromagnetic field at the point $\vec{x}$ is given by

$$\vec{l}(\vec{x},t) = \vec{x} \times \vec{p}(\vec{x},t) = \frac{1}{c^2}\vec{x} \times \vec{S}(\vec{x},t) = \frac{1}{4\pi c}\vec{x} \times [\vec{E}(\vec{x},t) \times \vec{B}(\vec{x},t)]. \tag{6.3.12}$$

One can show that, as a consequence of Maxwell's equations, energy, momentum, and angular momentum are all conserved.

6.4 Reflection of an Electromagnetic Wave

Let us consider a perfect conductor with a planar surface in the x_1-x_2-plane at $x_3 = 0$. This surface acts as a perfect mirror for incident electromagnetic radiation. For simplicity we consider an incident plane wave with wave vector $\vec{k}_i$ and angular frequency ω_i. Without loss of generality, we can assume that $\vec{k}_i$ is in the x_1-x_3-plane. Now we decompose the electric field into the 1- and 3-components in the plane and the 2-component out of the plane. First, we consider the in-plane components. The corresponding component of the magnetic field is then in the 2-direction and we may write

$$\vec{E}_i(\vec{x},t) = \vec{E}_{0i}\cos(\vec{k}_i \cdot \vec{x} - \omega_i t),\ \vec{B}_i(\vec{x},t) = -E_{0i}\cos(\vec{k}_i \cdot \vec{x} - \omega_i t)\vec{e}_y, \tag{6.4.1}$$

with $\vec{E}_{0i}$ being in the 1-3-plane. The incident wave will be reflected and then takes the form

$$\vec{E}_r(\vec{x},t) = \vec{E}_{0r}\cos(\vec{k}_r \cdot \vec{x} - \omega_r t + \varphi),\ \vec{B}_r(\vec{x},t) = -E_{0r}\cos(\vec{k}_r \cdot \vec{x} - \omega_r t + \varphi)\vec{e}_y, \tag{6.4.2}$$

and the total electromagnetic field is given by

$$\vec{E}(\vec{x},t) = \vec{E}_i(\vec{x},t) + \vec{E}_r(\vec{x},t),\ \vec{B}(\vec{x},t) = \vec{B}_i(\vec{x},t) + \vec{B}_r(\vec{x},t). \tag{6.4.3}$$

Now we want to determine $\vec{E}_{0r}$, $\vec{k}_r$, ω_r, and φ in terms of $\vec{E}_{0i}$, $\vec{k}_i$, and ω_i. We use the boundary conditions for the total electromagnetic field

$$\vec{n} \times \vec{E} = 0,\ \vec{n} \cdot \vec{B} = 0. \tag{6.4.4}$$

The second equation is automatically satisfied because $\vec{B}$ points in the 2-direction. As a consequence of the first equation the tangential 1-component of $\vec{E}$ vanishes at the mirror (i.e. at $z = 0$)

$$E_{0i} \cos\theta_i \cos(k_{i1}x_1 - \omega_i t) - E_{0r} \cos\theta_r \cos(k_{r1}x_1 - \omega_r t + \varphi) = 0. \quad (6.4.5)$$

This equation must be valid for all x_1 and t which implies

$$E_{0i} \cos\theta_i = E_{0r} \cos\theta_r, \ k_{i1} = k_{r1}, \ \omega_i = \omega_r, \ \varphi = 0. \quad (6.4.6)$$

In particular, the boundary conditions imply that the angles θ_i and θ_r are the same, as we know, of course, from our experience with mirrors. It is a good exercise to repeat these considerations for the other components of the wave with $\vec{E}$ pointing in the 2-direction.

6.5 A Coaxial Transmission Line

Transmission lines are technical devices consisting of two (or more) conducting surfaces that are used to transport electromagnetic waves in a given direction. A parallel plate geometry is simple to compute but not practical. A coaxial geometry, on the other hand, is practical, but more complicated to analyze mathematically. The geometry is as follows. There is an outer conductor of radius a and an inner conductor of radius b. The system has a cylindrical symmetry. Hence, it is useful to work with cylindrical coordinates ρ, φ, and z. Here ρ measures the distance from the z-axis, φ measures the angle with the x-axis, and z is the ordinary z-coordinate. The three unit-vectors we are dealing with are

$$\vec{e}_\rho = (\cos\varphi, \sin\varphi, 0), \ \vec{e}_\varphi = (-\sin\varphi, \cos\varphi, 0), \ \vec{e}_z = (0, 0, 1). \quad (6.5.1)$$

Again, we need to satisfy the boundary conditions $\vec{n} \times \vec{E} = 0$ and $\vec{n} \cdot \vec{B} = 0$ at the surface of the two conductors. Here the vector normal to the surface is just $\vec{n} = \vec{e}_\rho$. Since we are interested in an electromagnetic wave propagating in the z-direction, we should have $\vec{E}$ and $\vec{B}$ fields perpendicular to $\vec{e}_z$, i.e. $\vec{e}_z \cdot \vec{E} = 0$ and $\vec{e}_z \cdot \vec{B} = 0$. Hence, $\vec{E}$ must point radially (along $\vec{e}_\rho$) and $\vec{B}$ must point tangentially (along $\vec{e}_\varphi$). Thus we make the following ansatz

$$\vec{E}(\vec{x}, t) = E(\rho) \cos(kz - \omega t)\vec{e}_\rho, \ \vec{B}(\vec{x}, t) = B(\rho) \cos(kz - \omega t)\vec{e}_\varphi. \quad (6.5.2)$$

Of course, a priori it is not obvious that such an ansatz will indeed solve Maxwell's equations. In order to verify this, we need to compute the divergence and the curl of a vector field

$$\vec{A} = A_\rho \vec{e}_\rho + A_\varphi e_\varphi + A_z \vec{e}_z \quad (6.5.3)$$

in cylindrical coordinates. According to appendix C, these are given by

$$\begin{aligned}\vec{\nabla}\cdot\vec{A} &= \frac{1}{\rho}\partial_\rho(\rho A_\rho) + \frac{1}{\rho}\partial_\varphi A_\varphi + \partial_z A_z,\\ \vec{\nabla}\times\vec{A} &= (\frac{1}{\rho}\partial_\varphi A_z - \partial_z A_\varphi)\vec{e}_\rho + (\partial_z A_\rho - \partial_\rho A_z)\vec{e}_\varphi\\ &+ (\frac{1}{\rho}\partial_\rho(\rho A_\rho) - \frac{1}{\rho}\partial_\varphi A_\rho)\vec{e}_z. \end{aligned} \tag{6.5.4}$$

Our ansatz for $\vec{E}$ and $\vec{B}$ implies

$$\begin{aligned}&E_\rho = E(\rho)\cos(kz-\omega t),\ E_\varphi = 0,\ E_z = 0,\\ &B_\rho = 0,\ B_\varphi = B(\rho)\cos(kz-\omega t),\ B_z = 0.\end{aligned} \tag{6.5.5}$$

Hence, we find

$$\begin{aligned}\vec{\nabla}\cdot\vec{E} &= \frac{1}{\rho}\partial_\rho[\rho E(\rho)\cos(kz-\omega t)] = \frac{1}{\rho}\cos(kz-\omega t)\partial_\rho[\rho E(\rho)],\\ \vec{\nabla}\times\vec{E} &= \partial_z[E(\rho)\cos(kz-\omega t)]\vec{e}_\varphi = -kE(\rho)\sin(kz-\omega t)\vec{e}_\varphi,\\ \vec{\nabla}\cdot\vec{B} &= \frac{1}{\rho}\partial_\varphi[B(\rho)\cos(kz-\omega t)] = 0,\\ \vec{\nabla}\times\vec{B} &= -\partial_z[B(\rho)\cos(kz-\omega t)\vec{e}_\rho + \frac{1}{\rho}\partial_\rho[\rho B(\rho)\cos(kz-\omega t)]\vec{e}_z\\ &= kB(\rho)\sin(kz-\omega t)\vec{e}_\rho + \frac{1}{\rho}\cos(kz-\omega t)\partial_\rho[\rho B(\rho)]\vec{e}_z.\end{aligned} \tag{6.5.6}$$

The third equation is automatically satisfied. In order to check the other Maxwell equations we also need

$$\partial_t\vec{E} = E(\rho)\omega\sin(kz-\omega t)\vec{e}_\rho,\ \partial_t\vec{B} = B(\rho)\omega\sin(kz-\omega t)\vec{e}_\varphi. \tag{6.5.7}$$

Let us assume that there is a vacuum between the two conductors. The various Maxwell equations then imply

$$\begin{aligned}&\vec{\nabla}\cdot\vec{E} = 0\ \Rightarrow\ \partial_\rho[\rho E(\rho)] = 0\ \Rightarrow\ E(\rho) = \frac{E_0}{\rho},\\ &\vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0,\ \Rightarrow\ -kE(\rho) + \frac{\omega}{c}B(\rho) = 0\ \Rightarrow\ B(\rho) = \frac{kc}{\omega}E(\rho),\\ &\vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = 0,\ \Rightarrow\ B(\rho) = \frac{\omega}{kc}E(\rho),\\ &\partial_\rho[\rho B(\rho)] = 0\ \Rightarrow B(\rho) = \frac{B_0}{\rho}.\end{aligned} \tag{6.5.8}$$

As a consequence of Maxwell's equations we thus obtain

$$\omega = kc, \ B(\rho) = E(\rho) = \frac{E_0}{\rho}, \tag{6.5.9}$$

exactly as for waves propagating in infinite space. In particular, the non-planar waves in the transmission line still travel at the same speed c. The arbitrary constant E_0 determines the intensity of the wave.

Let us also compute the energy flux, i.e. the Poynting vector

$$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B} = \frac{c}{4\pi}\frac{E_0^2}{\rho^2}\cos^2(kz - \omega t)\vec{e}_z. \tag{6.5.10}$$

As expected, the energy flows in the z-direction. The energy flux decays proportional to $1/\rho^2$ as we move away from the symmetry axis of the transmission line.

The propagating electromagnetic wave induces charges and currents at the surface of the two conductors. Let us consider the inner conductor. Using

$$\vec{n} \cdot \vec{E} = 4\pi\rho_s, \ \vec{n} \times \vec{B} = \frac{4\pi}{c}\vec{j}_s, \tag{6.5.11}$$

at the surface and using $\vec{n} = \vec{e}_\rho$ we find

$$\rho_s = \frac{1}{4\pi}\frac{E_0}{b}\cos(kz - \omega t), \ \vec{j}_s = \frac{c}{4\pi}\frac{E_0}{b}\cos(kz - \omega t)\vec{e}_z. \tag{6.5.12}$$

Now we can compute the total induced surface charge per length l

$$\frac{Q}{l} = 2\pi b \rho_s = \frac{E_0}{2}\cos(kz - \omega t), \tag{6.5.13}$$

as well as the total current

$$I = 2\pi b j_{s3} = \frac{E_0}{2c}\cos(kz - \omega t). \tag{6.5.14}$$

Chapter 7

Structure of the Wave Equation

Until now we have considered the propagation of electromagnetic waves in a vacuum without considering how these waves were generated. As we have already seen in electro- and magnetostatics, time-independent charges and currents are the sources of static electric and magnetic field. As we will see in this chapter, time-varying charges and currents lead to the radiation of electromagnetic waves. We also study the structure of the wave equation, in particular its relativistic nature as well as the issue of time-reversal.

7.1 Wave Equation with Charge and Current Densities

Let us consider Maxwell's equations in the presence of charges and currents

$$\begin{aligned}
&\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = 4\pi\rho(\vec{x}, t), \\
&\vec{\nabla} \times \vec{E}(\vec{x}, t) + \frac{1}{c}\partial_t \vec{B}(\vec{x}, t) = 0, \\
&\vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0, \\
&\vec{\nabla} \times \vec{B}(\vec{x}, t) - \frac{1}{c}\partial_t \vec{E}(\vec{x}, t) = \frac{4\pi}{c}\vec{j}(\vec{x}, t),
\end{aligned} \tag{7.1.1}$$

and let us express the electromagnetic field in terms of scalar and vector potentials as

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\Phi(\vec{x},t)' - \frac{1}{c}\partial_t \vec{A}(\vec{x},t)',$$
$$\vec{B}(\vec{x},t) = \vec{\nabla} \times \vec{A}(\vec{x},t)'. \tag{7.1.2}$$

As we have seen earlier, the homogeneous Maxwell equations are then automatically satisfied because

$$\vec{\nabla} \times \vec{E}(\vec{x},t) + \frac{1}{c}\partial_t \vec{B}(\vec{x},t) =$$
$$-\vec{\nabla} \times \vec{\nabla} \cdot \Phi(\vec{x},t)' - \frac{1}{c}\vec{\nabla} \times \partial_t \vec{A}(\vec{x},t)' + \frac{1}{c}\partial_t \vec{\nabla} \times \vec{A}(\vec{x},t)' = 0,$$
$$\vec{\nabla} \cdot \vec{B}(\vec{x},t) = \vec{\nabla} \cdot \vec{\nabla} \times \vec{A}(\vec{x},t)' = 0, \tag{7.1.3}$$

The inhomogeneous Maxwell equations, on the other hand, take the form

$$\begin{aligned}
\vec{\nabla} \cdot \vec{E}(\vec{x},t) &= -\Delta\Phi(\vec{x},t)' - \frac{1}{c}\partial_t \vec{\nabla} \cdot \vec{A}(\vec{x},t)' = 4\pi\rho(\vec{x},t), \\
\vec{\nabla} \times \vec{B}(\vec{x},t) - \frac{1}{c}\partial_t \vec{E}(\vec{x},t) &= -\Delta\vec{A}(\vec{x},t)' + \vec{\nabla}\vec{\nabla} \cdot \vec{A}(\vec{x},t)' \\
&+ \frac{1}{c}\partial_t \vec{\nabla}\Phi(\vec{x},t)' + \frac{1}{c^2}\partial_t^2 \vec{A}(\vec{x},t)' = \frac{4\pi}{c}\vec{j}(\vec{x},t).
\end{aligned} \tag{7.1.4}$$

In the next step we impose the Lorentz gauge condition

$$\vec{\nabla} \cdot \vec{A}(\vec{x},t)' + \frac{1}{c}\partial_t \Phi(\vec{x})' = 0, \tag{7.1.5}$$

which can be satisfied by performing a gauge transformation

$$\Phi(\vec{x},t)' = \Phi(\vec{x},t) + \frac{1}{c}\partial_t \varphi(\vec{x},t),$$
$$\vec{A}(\vec{x},t)' = \vec{A}(\vec{x},t) - \vec{\nabla}\varphi(\vec{x},t), \tag{7.1.6}$$

which leaves the electromagnetic field invariant and leads to

$$\vec{\nabla}\cdot\vec{A}(\vec{x},t)' + \frac{1}{c}\partial_t \Phi(\vec{x})' = \vec{\nabla}\cdot\vec{A}(\vec{x},t) + \frac{1}{c}\partial_t \Phi(\vec{x}) - \Delta\varphi(\vec{x},t) + \frac{1}{c^2}\partial_t^2 \varphi(\vec{x},t) = 0, \tag{7.1.7}$$

such that the gauge transformation function $\varphi(\vec{x},t)$ must satisfy

$$\Delta\varphi(\vec{x},t) - \frac{1}{c^2}\partial_t^2 \varphi(\vec{x},t) = \vec{\nabla} \cdot \vec{A}(\vec{x},t) + \frac{1}{c}\partial_t \Phi(\vec{x}). \tag{7.1.8}$$

In the Lorentz gauge the inhomogeneous Maxwell equations decouple and simplify to

$$-\Delta\Phi(\vec{x},t)' + \frac{1}{c^2}\partial_t^2\Phi(\vec{x},t)' = 4\pi\rho(\vec{x},t),$$
$$-\Delta\vec{A}(\vec{x},t)' + \frac{1}{c^2}\partial_t^2\vec{A}(\vec{x},t)' = \frac{4\pi}{c}\vec{j}(\vec{x},t). \qquad (7.1.9)$$

7.2 Wave Equation in Relativistic Form

The inhomogeneous Maxwell equations in Lorentz gauge have the form of a wave equation

$$-\Delta A^\mu(x)' + \frac{1}{c^2}\partial_t^2 A^\mu(x)' = \frac{4\pi}{c}j^\mu(x). \qquad (7.2.1)$$

Here we have defined a point $x = (\vec{x},t)$ in 4-dimensional space-time as well as two so-called 4-vectors

$$A^\mu(x)' = (\Phi(\vec{x},t)', A_1(\vec{x},t)', A_2(\vec{x},t)', A_3(\vec{x},t)'),$$
$$j^\mu(x) = (c\rho(\vec{x},t), j_1(\vec{x},t), j_2(\vec{x},t), j_3(\vec{x},t)), \qquad (7.2.2)$$

which provide a first glance at the fact that electrodynamics is invariant under Lorentz transformations rotating space- into time-coordinates. Also the space-time point x itself can be described by a 4-vector

$$x^\mu = (x_0, x_1, x_2, x_3) = (ct, x_1, x_2, x_3). \qquad (7.2.3)$$

Remarkably, time (multiplied by the velocity of light) acts as a fourth coordinate in space-time. Introducing two space-time versions of the Nabla operator

$$\partial_\mu = (\frac{1}{c}\partial_t, \partial_1, \partial_2, \partial_3), \ \partial^\mu = (\frac{1}{c}\partial_t, -\partial_1, -\partial_2, -\partial_3), \qquad (7.2.4)$$

one can write the Lorentz gauge condition as

$$\partial_\mu A^\mu(x)' = \sum_{\mu=1}^{4}\partial_\mu A^\mu(x)' = \frac{1}{c}\partial_t\Phi(\vec{x},t)' + \vec{\nabla}\cdot\vec{A}(\vec{x})' = 0. \qquad (7.2.5)$$

Here we have used Einstein's summation convention according to which repeated indices are summed over. Similarly, the continuity equation which expresses electric charge conservation takes the form

$$\partial_\mu j^\mu(x) = \frac{1}{c}\partial_t \ c\rho(\vec{x},t) + \vec{\nabla}\cdot\vec{j}(\vec{x},t) = 0. \qquad (7.2.6)$$

In relativistic notation a gauge transformation takes the form

$$A^\mu(x)' = A^\mu(x) + \partial^\mu \varphi(x). \tag{7.2.7}$$

The relativistic version of the Laplacian is the so-called d'Alembertian

$$\Box = \partial_\mu \partial^\mu = \frac{1}{c^2}\partial_t^2 - \vec{\nabla}\cdot\vec{\nabla} = \frac{1}{c^2}\partial_t^2 - \Delta, \tag{7.2.8}$$

such that the wave equation in Lorentz gauge can be written as

$$\Box A^\mu(x) = \frac{4\pi}{c} j^\mu(x). \tag{7.2.9}$$

We will later return to the fascinating relativistic structure of electrodynamics and, in fact, of all of fundamental physics. For the rest of this chapter we return to a notation not using 4-vectors.

7.3 Wave Equation in Fourier Space

The first of the wave equations in Lorentz gauge is given by

$$\frac{1}{c^2}\partial_t^2 \Phi(\vec{x},t)' - \Delta\Phi(\vec{x},t)' = 4\pi\rho(\vec{x},t), \tag{7.3.1}$$

and the other three are analogous. In order to solve the above wave equation, let us introduce the Fourier transformations

$$\begin{aligned}
\Phi(\vec{k},\omega)' &= \int d^3x\ dt\ \Phi(\vec{x},t)' \exp(-i\vec{k}\cdot\vec{x} + i\omega t), \\
\rho(\vec{k},\omega) &= \int d^3x\ dt\ \rho(\vec{x},t) \exp(-i\vec{k}\cdot\vec{x} + i\omega t).
\end{aligned} \tag{7.3.2}$$

The inverse Fourier transforms then take the form

$$\begin{aligned}
\Phi(\vec{x},t)' &= \frac{1}{(2\pi)^4}\int d^3k\ d\omega\ \Phi(\vec{k},\omega)' \exp(i\vec{k}\cdot\vec{x} - i\omega t), \\
\rho(\vec{x},t) &= \frac{1}{(2\pi)^4}\int d^3k\ d\omega\ \rho(\vec{k},\omega) \exp(i\vec{k}\cdot\vec{x} - i\omega t).
\end{aligned} \tag{7.3.3}$$

Inserting this in eq.(7.3.1) we obtain

$$-\frac{\omega^2}{c^2}\Phi(\vec{k},\omega)' + \vec{k}^2\Phi(\vec{k},\omega)' = 4\pi\rho(\vec{k},\omega), \tag{7.3.4}$$

such that

$$\Phi(\vec{k},\omega)' = \frac{4\pi\rho(\vec{k},\omega)}{\vec{k}^2 - k_0^2}. \tag{7.3.5}$$

Here we have introduced $k_0 = \omega/c$ which can be viewed as the time-component of another 4-vector

$$k_\mu = (k_0, -\vec{k}), \tag{7.3.6}$$

such that

$$k_\mu x^\mu = k_0 x_0 - \vec{k}\cdot\vec{x} = \frac{\omega}{c} ct - \vec{k}\cdot\vec{x} = \omega t - \vec{k}\cdot\vec{x}. \tag{7.3.7}$$

Remarkably, the original partial differential equation has turned into a simple algebraic equation in Fourier space. However, we still need to return to coordinate space. For this purpose, we first consider the Green function method.

7.4 Retarded Green Function for the Wave Equation

Let us study the wave equation for the so-called Green function $G(\vec{x},t)$ which takes the form

$$\frac{1}{c^2}\partial_t^2 G(\vec{x},t) - \Delta G(\vec{x},t) = 4\pi\delta(\vec{x})\delta(t). \tag{7.4.1}$$

Once the Green function is known in coordinate space, the solution of the full wave equation is then obtained as

$$\Phi(\vec{x},t) = \int d^3x'\ dt'\ \rho(\vec{x}',t')G(\vec{x}'-\vec{x},t'-t), \tag{7.4.2}$$

because then indeed

$$[\frac{1}{c^2}\partial_t^2 - \Delta]\Phi(\vec{x},t) = \int d^3x'\ dt'\ \rho(\vec{x}',t')[\frac{1}{c^2}\partial_t^2 - \Delta]G(\vec{x}'-\vec{x},t'-t) = \\ \int d^3x'\ dt'\ \rho(\vec{x}',t')4\pi\delta(\vec{x}'-\vec{x})\delta(t'-t) = 4\pi\rho(\vec{x},t). \tag{7.4.3}$$

Since the Fourier transformation of a δ-function is simply given by

$$\delta(\vec{k},\omega) = \int d^3x\ dt\ \delta(\vec{x})\delta(t)\exp(-i\vec{k}\cdot\vec{x} + i\omega t) = 1, \tag{7.4.4}$$

in Fourier space the equation for the Green function takes the form

$$-\frac{\omega^2}{c^2}G(\vec{k},\omega) + \vec{k}^2 G(\vec{k},\omega) = 4\pi \ \Rightarrow\ G(\vec{k},\omega) = \frac{4\pi}{\vec{k}^2 - k_0^2}. \tag{7.4.5}$$

As before, we need to know the form of the Green function in coordinate space. Let us verify that in coordinate space the so-called retarded Green function takes the form

$$G_r(\vec{x},t) = \frac{1}{|\vec{x}|}\delta(t + |\vec{x}|/c). \tag{7.4.6}$$

For this purpose we compute its Fourier transform

$$\begin{aligned}
G_r(\vec{k},\omega) &= \int d^3x\ dt\ G_r(\vec{x},t)\exp(-i\vec{k}\cdot\vec{x} + i\omega t), \\
&= \int d^3x\ dt\ \frac{\exp(-i\vec{k}\cdot\vec{x} + i\omega t)}{|\vec{x}|}\delta(t + |\vec{x}|/c) \\
&= \int d^3x \frac{\exp(-i\vec{k}\cdot\vec{x} - i\omega|\vec{x}|/c)}{|\vec{x}|} \\
&= 2\pi\int_0^\infty dr r^2 \int_{-1}^1 d\cos\theta\ \frac{\exp(-i|\vec{k}|r\cos\theta - ik_0 r)}{r} \\
&= 4\pi\int_0^\infty dr \frac{\sin(|\vec{k}|r)}{|\vec{k}|}\exp(-ik_0 r).
\end{aligned} \tag{7.4.7}$$

This integral is ill-defined for real values of k_0. However, it can be made well-defined by introducing a convergence generating factor ε which makes k_0 complex but is finally sent to zero, i.e.

$$\begin{aligned}
G_r(\vec{k},\omega) &= \lim_{\varepsilon\to 0} 4\pi\int_0^\infty dr\frac{\sin(|\vec{k}|r)}{|\vec{k}|}\exp(-i(k_0 - i\varepsilon)r) \\
&= \lim_{\varepsilon\to 0}\frac{4\pi}{\vec{k}^2 - (k_0 - i\varepsilon)^2}.
\end{aligned} \tag{7.4.8}$$

In the limit $\varepsilon \to 0$ this indeed gives the correct form of the Fourier transform $G(\vec{k},\omega)$. Inserting eq.(7.4.6) into eq.(7.4.2) one obtains the so-called retarded solution of the wave equation

$$\begin{aligned}
\Phi_r(\vec{x},t)' &= \int d^3x'\ dt'\ \rho(\vec{x}',t')G_r(\vec{x}' - \vec{x}, t' - t) \\
&= \int d^3x'\ dt'\ \rho(\vec{x}',t')\frac{1}{|\vec{x}' - \vec{x}|}\delta(t' - t + |\vec{x}' - \vec{x}|/c) \\
&= \int d^3x'\ \frac{\rho(\vec{x}', t - |\vec{x}' - \vec{x}|/c)}{|\vec{x}' - \vec{x}|}.
\end{aligned} \tag{7.4.9}$$

For a time-independent charge density this reduces to the equation

$$\Phi(\vec{x},t)' = \int d^3x'\ \frac{\rho(\vec{x}')}{|\vec{x}' - \vec{x}|}, \tag{7.4.10}$$

which we know from electrostatics. For a time-dependent charge density, on the other hand, there are retardation effects. In particular, the field $\Phi(\vec{x},t)'$ at position $\vec{x}$ at time t is influenced by the charge density $\rho(\vec{x}',t')$ at position $\vec{x}'$ at time

$$t' = t - |\vec{x}' - \vec{x}|/c. \tag{7.4.11}$$

The time-delay or retardation $t - t'$ thus corresponds to the time a light signal (of velocity c) needs to travel the distance $|\vec{x}' - \vec{x}|$. This is consistent with what we know about the propagation of electromagnetic radiation. A radio broadcast reaches us slightly later than it was aired from the antenna of the radio station.

The wave equations for the vector potential can be solved in a completely analogous way which leads to

$$\vec{A}_r(\vec{x},t)' = \frac{1}{c}\int d^3x' \ \frac{\vec{j}(\vec{x}', t - |\vec{x}' - \vec{x}|/c)}{|\vec{x}' - \vec{x}|}. \tag{7.4.12}$$

Again, for a time-independent current density this implies

$$\vec{A}(\vec{x})' = \frac{1}{c}\int d^3x' \ \frac{\vec{j}(\vec{x}')}{|\vec{x}' - \vec{x}|}, \tag{7.4.13}$$

which is an equation that we already encountered in magnetostatics.

7.5 Advanced Green Function and Causality

There is another so-called advanced Green function

$$G_a(\vec{x},t) = \frac{1}{|\vec{x}|}\delta(t - |\vec{x}|/c). \tag{7.5.1}$$

Its Fourier transform is given by

$$\begin{aligned} G_a(\vec{k},\omega) &= \int d^3x \ dt \ G_a(\vec{x},t) \exp(-i\vec{k}\cdot\vec{x} + i\omega t), \\ &= \int d^3x \ dt \ \frac{\exp(-i\vec{k}\cdot\vec{x} + i\omega t)}{|\vec{x}|}\delta(t - |\vec{x}|/c) \\ &= \int d^3x \frac{\exp(-i\vec{k}\cdot\vec{x} + i\omega|\vec{x}|/c)}{|\vec{x}|} \\ &= 2\pi\int_0^\infty drr^2\int_{-1}^1 d\cos\theta \ \frac{\exp(-i|\vec{k}|r\cos\theta - ik_0 r)}{r} \\ &= 4\pi\int_0^\infty dr\frac{\sin(|\vec{k}|r)}{|\vec{k}|}\exp(ik_0 r). \end{aligned} \tag{7.5.2}$$

To make this expression well-defined we introduce

$$\begin{aligned} G_a(\vec{k},\omega) &= \lim_{\varepsilon \to 0} 4\pi \int_0^\infty dr \frac{\sin(|\vec{k}|r)}{|\vec{k}|} \exp(i(k_0 + i\varepsilon)r) \\ &= \lim_{\varepsilon \to 0} \frac{4\pi}{\vec{k}^2 - (k_0 + i\varepsilon)^2}, \end{aligned} \tag{7.5.3}$$

which again gives the correct Fourier transform $G(\vec{k},\omega)$ in the limit $\varepsilon \to 0$. Inserting eq.(7.5.1) into eq.(7.4.2) one obtains an advanced solution of the wave equation

$$\begin{aligned} \Phi_a(\vec{x},t)' &= \int d^3x' \ dt' \ \rho(\vec{x}',t') G_a(\vec{x}' - \vec{x}, t' - t) \\ &= \int d^3x' \ dt' \ \rho(\vec{x}',t') \frac{1}{|\vec{x}' - \vec{x}|} \delta(t' - t - |\vec{x}' - \vec{x}|/c) \\ &= \int d^3x' \ \frac{\rho(\vec{x}', t + |\vec{x}' - \vec{x}|/c)}{|\vec{x}' - \vec{x}|}. \end{aligned} \tag{7.5.4}$$

This mathematical solution of the wave equation seems to make no physical sense. In particular, the potential $\Phi(\vec{x},t)'$ at time t is now related to the charge density at the later time

$$t' = t + |\vec{x}' - \vec{x}|/c. \tag{7.5.5}$$

This seems to violate causality. For example, it is as if the listeners of a radio program would receive electromagnetic waves coming from outer space, which would then stream into the radio station from all directions and are finally completely absorbed there. This is indeed what would happen if we would "film" the real world and play the "movie" backwards. As we will confirm in the next section, this paradoxical situation is indeed a mathematically correct solution of the wave equation, which is, however, not physically realized.

Remarkably, in the real world there is a preferred direction of time. It never happens that we receive a radio program before it is aired at the radio station. This has something to do with the second law of thermodynamics according to which entropy never decreases. It also has something to do with the initial conditions of our Universe. If Nature had decided to air an interesting radio program (played in reverse) in the moment of the big bang, we would now listen to this program shortly before it is completely absorbed by the radio station. Fortunately, the Universe does not confuse us that much. The "program" that was aired in the moment of the big bang is the cosmic background radiation. This radiation contains extremely valuable information about the beginning of our world and reaches us in a completely causal fashion.

7.6 Time-Reversal

If we could prepare the appropriate initial conditions, any electromagnetic phenomenon could be run backwards in time. In other words, what we see when playing the "movie" backwards is consistent with the laws of Nature. In more technical terms, time-reversal is a symmetry of Maxwell's equations. Let us convince ourselves that Maxwell's equations are indeed invariant under time-reversal. First of all, let us define the time-reversed charge and current densities

$$^T\rho(\vec{x},t) = \rho(\vec{x},-t), \ ^T\vec{j}(\vec{x},t) = -\vec{j}(\vec{x},-t). \tag{7.6.1}$$

We have changed the sign of the current density, because when playing the "movie" backwards, moving charges change their direction of motion. We also have

$$^T\Phi(\vec{x},t) = \Phi(\vec{x},-t), \ ^T\vec{A}(\vec{x},t) = -\vec{A}(\vec{x},-t), \tag{7.6.2}$$

such that

$$\begin{aligned} ^T\vec{E}(\vec{x},t) &= -\vec{\nabla}^T\Phi(\vec{x},t) - \frac{1}{c}\partial_t^T\vec{A}(\vec{x},t) = -\vec{\nabla}\Phi(\vec{x},-t) + \frac{1}{c}\partial_t\vec{A}(\vec{x},-t) \\ &= -\vec{\nabla}\Phi(\vec{x},-t) - \frac{1}{c}[\partial_t\vec{A}](\vec{x},-t) = \vec{E}(\vec{x},-t), \\ ^T\vec{B}(\vec{x},t) &= \vec{\nabla}\times^T\vec{A}(\vec{x},t) = -\vec{\nabla}\times\vec{A}(\vec{x},-t) = -\vec{B}(\vec{x},-t). \end{aligned} \tag{7.6.3}$$

Let us now check time-reversal invariance of the individual Maxwell equations. We find

$$\begin{aligned} &\vec{\nabla}\cdot^T\vec{E}(\vec{x},t) = \vec{\nabla}\cdot\vec{E}(\vec{x},-t) = 4\pi\rho(\vec{x},-t) = 4\pi^T\rho(\vec{x},t), \\ &\vec{\nabla}\times^T\vec{E}(\vec{x},t) + \frac{1}{c}\partial_t^T\vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{E}(\vec{x},-t) - \frac{1}{c}\partial_t\vec{B}(\vec{x},-t) = \\ &\vec{\nabla}\times\vec{E}(\vec{x},-t) + \frac{1}{c}[\partial_t\vec{B}](\vec{x},-t) = 0, \\ &\vec{\nabla}\cdot^T\vec{B}(\vec{x},t) = -\vec{\nabla}\cdot\vec{B}(\vec{x},-t) = 0, \\ &\vec{\nabla}\times^T\vec{B}(\vec{x},t) - \frac{1}{c}\partial_t^T\vec{E}(\vec{x},t) = -\vec{\nabla}\times\vec{B}(\vec{x},-t) - \frac{1}{c}\partial_t\vec{E}(\vec{x},-t) = \\ &-\vec{\nabla}\times\vec{B}(\vec{x},-t) + \frac{1}{c}[\partial_t\vec{E}](\vec{x},-t) = -\frac{4\pi}{c}\vec{j}(\vec{x},-t) = \frac{4\pi}{c}{}^T\vec{j}(\vec{x},t). \end{aligned} \tag{7.6.4}$$

Hence, we see explicitly that the time-reversed fields (the ones we "see" when playing the "movie" backwards) also satisfy Maxwell's equations and are hence consistent with the laws of Nature, although it may be difficult to create the appropriate initial conditions. Let us now apply time-reversal to the retarded solution, i.e.

$$\begin{aligned} {}^T\Phi_r(\vec{x},t)' &= \Phi_r(\vec{x},-t)' = \int d^3x' \, \frac{\rho(\vec{x}',-t-|\vec{x}'-\vec{x}|/c)}{|\vec{x}'-\vec{x}|} \\ &= \int d^3x' \, \frac{{}^T\rho(\vec{x}',t+|\vec{x}'-\vec{x}|/c)}{|\vec{x}'-\vec{x}|}. \end{aligned} \tag{7.6.5}$$

This is indeed the advanced solution corresponding to the time-reversed charge density.

Let us also consider time-reversal applied to the Lorentz force

$$\vec{F}(t) = q[\vec{E}((\vec{r}(t),t) + \frac{\vec{v}(t)}{c} \times \vec{B}(\vec{r}(t),t)], \tag{7.6.6}$$

acting on a charged particle at position $\vec{r}(t)$ moving with the velocity $\vec{v}(t) = d\vec{r}(t)/dt$. The time-reversed position and velocity are given by

$${}^T\vec{r}(t) = \vec{r}(-t) \;\Rightarrow\; {}^T\vec{v}(t) = \frac{d^T\vec{r}(t)}{dt} = \frac{d\vec{r}(-t)}{dt} = -\vec{v}(-t), \tag{7.6.7}$$

and hence the time-reversed Lorentz force is given by

$$\begin{aligned} {}^T\vec{F}(t) &= q[{}^T\vec{E}({}^T\vec{r}(t),t) + \frac{{}^T\vec{v}(t)}{c} \times^T \vec{B}({}^T\vec{r}(t),t)] \\ &= q[\vec{E}(\vec{r}(-t),-t) + \frac{\vec{v}(-t)}{c} \times \vec{B}(\vec{r}(-t),-t)] = \vec{F}(-t). \end{aligned} \tag{7.6.8}$$

It is important to note that classical mechanics, i.e. Newton's equation $\vec{F}(t) = m\vec{a}(t)$, is also time-reversal invariant. This follows immediately from the time-reversed acceleration

$${}^T\vec{a}(t) = \frac{d^T\vec{v}(t)}{dt} = -\frac{d\vec{v}(-t)}{dt} = \vec{a}(-t), \tag{7.6.9}$$

which indeed implies

$${}^T\vec{F}(t) = \vec{F}(-t) = m\vec{a}(-t) = m^T\vec{a}(t). \tag{7.6.10}$$

7.7 Parity and Charge Conjugation

Another symmetry that is worth discussing is parity, i.e. the inversion of all three spatial coordinates. We will find that Maxwell's equations are indeed parity-invariant. The transformed charge and current densities are

$${}^P\rho(\vec{x},t) = \rho(-\vec{x},t), \; {}^P\vec{j}(\vec{x},t) = -\vec{j}(-\vec{x},t), \tag{7.7.1}$$

and we also have

$$^P\Phi(\vec{x},t) = \Phi(-\vec{x},t), \ ^P\vec{A}(\vec{x},t) = -\vec{A}(-\vec{x},t), \tag{7.7.2}$$

such that

$$\begin{aligned}
^P\vec{E}(\vec{x},t) &= -\vec{\nabla}^P\Phi(\vec{x},t) - \frac{1}{c}\partial_t^P\vec{A}(\vec{x},t) = -\vec{\nabla}\Phi(-\vec{x},t) + \frac{1}{c}\partial_t\vec{A}(-\vec{x},t) \\
&= [\vec{\nabla}\Phi](-\vec{x},t) + \frac{1}{c}\partial_t\vec{A}(-\vec{x},t) = -\vec{E}(-\vec{x},t), \\
^P\vec{B}(\vec{x},t) &= \vec{\nabla}\times^P\vec{A}(\vec{x},t) = -\vec{\nabla}\times\vec{A}(-\vec{x},t) = [\vec{\nabla}\times\vec{A}](-\vec{x},t) \\
&= \vec{B}(-\vec{x},t).
\end{aligned} \tag{7.7.3}$$

Unlike the vectors $\vec{j}(\vec{x},t)$ and $\vec{E}(\vec{x},t)$ the magnetic field does not change sign under the parity-reflection. Indeed, $\vec{B}(\vec{x},t)$ is therefore called a pseudo-vector. Checking the parity invariance of the Maxwell equations we indeed obtain

$$\begin{aligned}
&\vec{\nabla}\cdot^P\vec{E}(\vec{x},t) = -\vec{\nabla}\cdot\vec{E}(-\vec{x},t) = [\vec{\nabla}\cdot\vec{E}](-\vec{x},t) = \\
&4\pi\rho(-\vec{x},t) = 4\pi^P\rho(\vec{x},t), \\
&\vec{\nabla}\times^P\vec{E}(\vec{x},t) + \frac{1}{c}\partial_t^P\vec{B}(\vec{x},t) = -\vec{\nabla}\times\vec{E}(-\vec{x},t) + \frac{1}{c}\partial_t\vec{B}(-\vec{x},t) = \\
&[\vec{\nabla}\times\vec{E}](-\vec{x},t) + \frac{1}{c}\partial_t\vec{B}(-\vec{x},t) = 0, \\
&\vec{\nabla}\cdot^P\vec{B}(\vec{x},t) = \vec{\nabla}\cdot\vec{B}(-\vec{x},t) = 0, \\
&\vec{\nabla}\times^P\vec{B}(\vec{x},t) - \frac{1}{c}\partial_t^P\vec{E}(\vec{x},t) = \vec{\nabla}\times\vec{B}(-\vec{x},t) + \frac{1}{c}\partial_t\vec{E}(-\vec{x},t) = \\
&-[\vec{\nabla}\times\vec{B}](-\vec{x},t) + \frac{1}{c}\partial_t\vec{E}](-\vec{x},t) = \\
&-\frac{4\pi}{c}\vec{j}(-\vec{x},t) = \frac{4\pi}{c}{}^P\vec{j}(\vec{x},t),
\end{aligned} \tag{7.7.4}$$

i.e. the parity-reflected fields also satisfy Maxwell's equations.

Let us also consider the parity-reflected Lorentz force

$$\begin{aligned}
^P\vec{F}(t) &= q[^P\vec{E}(^P\vec{r}(t),t) + \frac{^P\vec{v}(t)}{c}\times^P\vec{B}(^P\vec{r}(t),t)] \\
&= q[-\vec{E}(-\vec{r}(t),t) - \frac{\vec{v}(t)}{c}\times\vec{B}(-\vec{r}(t),t)] = -\vec{F}(t),
\end{aligned} \tag{7.7.5}$$

which results from

$$^P\vec{r}(t) = -\vec{r}(t) \ \Rightarrow \ ^P\vec{v}(t) = \frac{d^P\vec{r}(t)}{dt} = -\frac{d\vec{r}(t)}{dt} = -\vec{v}(t). \tag{7.7.6}$$

Since we have

$$^P\vec{a}(t) = \frac{d^P\vec{v}(t)}{dt} = -\frac{d\vec{v}(t)}{dt} = -\vec{a}(t), \tag{7.7.7}$$

Newton's equation is also parity invariant

$$^P\vec{F}(t) = -\vec{F}(t) = -m\vec{a}(t) = m^P\vec{a}(t). \tag{7.7.8}$$

It should be noted that not all fundamental interactions are parity-symmetric. For example, the weak interactions (responsible, for example, for the decay of a neutron into a proton, an electron, and an anti-neutrino) explicitly violate the parity symmetry. This is because neutrinos have a definite handedness, and right- and left-handed neutrinos have different physical properties.

Another fundamental symmetry is charge conjugation. All elementary particles have anti-particles with the opposite charge. If we would replace all matter by anti-matter, the electromagnetic interactions would be as before as long as we substitute

$$\begin{aligned} &^C\rho(\vec{x},t) = -\rho(\vec{x},t),\ ^C\vec{j}(\vec{x},t) = -\vec{j}(\vec{x},t), \\ &^C\Phi(\vec{x},t) = -\Phi(\vec{x},t),\ ^C\vec{A}(\vec{x},t) = -\vec{A}(\vec{x},t), \end{aligned} \tag{7.7.9}$$

such that

$$\begin{aligned} &^C\vec{E}(\vec{x},t) = -\vec{\nabla}^C\Phi(\vec{x},t) - \frac{1}{c}\partial_t^C\vec{A}(\vec{x},t) = -\vec{E}(\vec{x},t), \\ &^C\vec{B}(\vec{x},t) = \vec{\nabla}\times^C\vec{A}(\vec{x},t) = -\vec{\nabla}\times\vec{A}(\vec{x},t) = -\vec{B}(\vec{x},t). \end{aligned} \tag{7.7.10}$$

Obviously, Maxwell's equations are invariant under a sign change of the electric and magnetic field as long as we change the sign of the charge and current density as well.

Finally, using $^Cq = -q$, let us also consider the charge conjugate of the Lorentz force

$$\begin{aligned} ^C\vec{F}(t) &= \ ^Cq[^C\vec{E}(\vec{r}(t),t) + \frac{\vec{v}(t)}{c}\times^C\vec{B}(\vec{r}(t),t)] \\ &= -q[-\vec{E}((\vec{r}(t),t) - \frac{\vec{v}(t)}{c}\times\vec{B}(\vec{r}(t),t)] = \vec{F}(t), \end{aligned} \tag{7.7.11}$$

which implies that Newton's equation is also charge conjugation invariant. Again, charge conjugation is not a symmetry of all fundamental interactions. In particular, the weak interactions violate not only P but also C, but they leave the combination CP invariant. Interestingly, even this is not true for all phenomena in Nature. In particular, there are extremely rare processes between unstable elementary particles mediated by the so-called Higgs field, which violate both CP and time-reversal T. Remarkably, one can prove the so-called CPT theorem which states that any relativistic local field theory is invariant under the reflection of all space-time coordinates PT combined with charge conjugation C. Only string theories, which are relativistic but to some extent non-local may eventually violate the CPT invariance.

7.8 Radiation from an Accelerated Charge

Let us consider a charge q at rest at the origin. Its electric field is given by

$$\vec{E}(\vec{x},t) = \frac{q}{|\vec{x}|^2}\vec{n}, \tag{7.8.1}$$

with $\vec{x} = |\vec{x}|\vec{n}$. Since the charge is at rest there is no magnetic field. Hence the Poynting vector vanishes and there is thus no energy flux. A charge at rest does not radiate electromagnetic waves. Similarly, a charge in uniform (unaccelerated) motion does not radiate either, because it appears to be at rest for a comoving observer. As we will see in this section, accelerated charges, however, do radiate electromagnetic waves.

Let us consider a charge q that moves around with position $\vec{r}(t)$, velocity $\vec{v}(t) = d\vec{r}(t)/dt$, and acceleration $\vec{a}(t) = d\vec{v}(t)/dt$. The corresponding charge and current densities then take the form

$$\rho(\vec{x},t) = q\delta(\vec{x}-\vec{r}(t)),\ \vec{j}(\vec{x},t) = q\vec{v}(t)\delta(\vec{x}-\vec{r}(t)). \tag{7.8.2}$$

using the expressions for the retarded potentials we now obtain

$$\begin{aligned}
\Phi_r(\vec{x},t)' &= \int d^3x' \frac{\rho(\vec{x}',t-|\vec{x}'-\vec{x}|/c)}{|\vec{x}'-\vec{x}|} \\
&= \int d^3x' \frac{q\delta(\vec{x}'-\vec{r}(t-|\vec{x}'-\vec{x}|/c))}{|\vec{x}'-\vec{x}|}, \\
\vec{A}_r(\vec{x},t)' &= \frac{1}{c}\int d^3x' \frac{\vec{j}(\vec{x}',t-|\vec{x}'-\vec{x}|/c)}{|\vec{x}'-\vec{x}|} \\
&= \frac{1}{c}\int d^3x' \frac{q\vec{v}(t-|\vec{x}'-\vec{x}|/c)\delta(\vec{x}'-\vec{r}(t-|\vec{x}'-\vec{x}|/c))}{|\vec{x}'-\vec{x}|}. \qquad (7.8.3)
\end{aligned}$$

These expressions are correct even for charges moving with relativistic velocities. Here we want to limit ourselves to small non-relativistic velocities with $\vec{v}(t) \ll c$.

Also we want to consider the radiation field far from the accelerated charge. In that case, one obtains

$$\Phi_r(\vec{x},t)' = \frac{q}{|\vec{x}|}\left(1 + \frac{1}{c}\vec{v}(t')\cdot\vec{n}\right), \ \vec{A}_r(\vec{x},t)' = \frac{q}{c|\vec{x}|}\vec{v}(t'), \tag{7.8.4}$$

with the retardation effect described by

$$t' = t - \frac{|\vec{x}|}{c}. \tag{7.8.5}$$

Let us now consider the electric and magnetic fields

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\Phi_r(\vec{x},t)' - \frac{1}{c}\partial_t\vec{A}_r(\vec{x},t)', \ \vec{B}(\vec{x},t) = \vec{\nabla}\times\vec{A}_r(\vec{x},t)'. \tag{7.8.6}$$

Keeping in mind that t' is a function of t and $\vec{x}$, we obtain

$$\partial_t\vec{v}(t') = \frac{d\vec{v}(t')}{dt'} = \vec{a}(t'), \ \partial_i\vec{v}(t') = \frac{d\vec{v}(t')}{dt'}\left(-\partial_i\frac{|\vec{x}|}{c}\right) = -\frac{1}{c}\vec{a}(t')\frac{x_i}{|\vec{x}|}, \tag{7.8.7}$$

which implies

$$\vec{\nabla}\cdot\vec{v}(t') = -\frac{1}{c}\vec{a}(t')\cdot\vec{n}, \ \vec{\nabla}\times\vec{v}(t') = \frac{1}{c}\vec{a}(t')\times\vec{n}. \tag{7.8.8}$$

We then find

$$\begin{aligned}
\vec{\nabla}\Phi_r(\vec{x},t)' &= -\frac{q}{|\vec{x}|^2}\vec{n}\left(1+\frac{1}{c}\vec{v}(t')\cdot\vec{n}\right) + \frac{q}{|\vec{x}|c}\vec{n}\left(-\frac{1}{c}\vec{a}(t')\cdot\vec{n}\right) + \frac{2q}{|\vec{x}|^2 c}\vec{v}(t'), \\
\partial_t\vec{A}_r(\vec{x},t)' &= \frac{q}{|\vec{x}|c}\vec{a}(t'), \\
\vec{\nabla}\times\vec{A}_r(\vec{x},t)' &= \frac{q\vec{v}(t')\times\vec{n}}{|\vec{x}|^2 c} + \frac{q\vec{a}(t')\times\vec{n}}{|\vec{x}|c^2}.
\end{aligned} \tag{7.8.9}$$

Neglecting terms that decay faster than $1/|\vec{x}|$ at large distances one obtains

$$\vec{E}(\vec{x},t) = \frac{q[\vec{a}(t')\times\vec{n}]\times\vec{n}}{|\vec{x}|c^2}, \ \vec{B}(\vec{x},t) = \frac{q\vec{a}(t')\times\vec{n}}{|\vec{x}|c^2}. \tag{7.8.10}$$

In particular, we have

$$\vec{E}(\vec{x},t) = \vec{B}(\vec{x},t)\times\vec{n}. \tag{7.8.11}$$

The electric field lies in the plane of the vectors $\vec{a}(t')$ and $\vec{n}$, and the magnetic field is perpendicular to both $\vec{n}$ and $\vec{E}$.

Let us now consider the Poynting vector

$$\begin{aligned}\vec{S}(\vec{x},t) &= \frac{c}{4\pi}\vec{E}(\vec{x},t)\times\vec{B}(\vec{x},t) = \frac{c}{4\pi}[\vec{B}(\vec{x},t)\times\vec{n}]\times\vec{B}(\vec{x},t) \\ &= \frac{c}{4\pi}|\vec{B}(\vec{x},t)|^2\vec{n} \\ &= \frac{q^2|\vec{a}(t')\times\vec{n}|^2}{4\pi c^3|\vec{x}|^2}\vec{n} = \frac{q^2|\vec{a}(t')|^2\sin^2\theta}{4\pi c^3|\vec{x}|^2}\vec{n}. \end{aligned} \tag{7.8.12}$$

Here θ is the angle between $\vec{n}$ and $\vec{a}(t')$. It should be noted that no power is radiated in the direction of $\vec{a}(t')$, and the maximum power is radiated at a 90 degrees angle. To compute the total power radiated by the charge, we integrate the Poynting flux through a closed sphere

$$\begin{aligned} P(t) &= 2\pi|\vec{x}|^2\int_0^\pi d\theta\sin\theta\frac{q^2|\vec{a}(t')|^2\sin^2\theta}{4\pi c^3|\vec{x}|^2} \\ &= \frac{q^2|\vec{a}(t')|^2}{2c^3}\int_{-1}^1 d\cos\theta(1-\cos^2\theta) = \frac{2q^2|\vec{a}(t')|^2}{3c^3}. \end{aligned} \tag{7.8.13}$$

This result is known as Larmor's formula.

7.9 Radiation from an Antenna

An antenna is a wire in which electric charges are accelerated and hence radiate electromagnetic waves. Let us consider a small antenna oriented in the z-direction with length l and an oscillating current

$$I(t) = I_0\cos(\omega t) \tag{7.9.1}$$

oscillating in it. The current corresponds to a moving charges with velocity

$$\vec{v}(t) = \frac{I_0 l}{q}\cos(\omega t)\vec{e}_z \;\Rightarrow\; q\vec{a}(t) = -I_0 l\omega\sin(\omega t)\vec{e}_z. \tag{7.9.2}$$

Next we use the previous equations for accelerated charges and we write

$$\begin{aligned}\vec{E}(\vec{x},t) &= \frac{I_0 l\omega}{c^2}\sin(\omega t')\frac{[\vec{e}_z\times\vec{n}]\times\vec{n}}{|\vec{x}|}, \\ \vec{B}(\vec{x},t) &= \frac{I_0 l\omega}{c^2}\sin(\omega t')\frac{\vec{e}_z\times\vec{n}}{|\vec{x}|}. \end{aligned} \tag{7.9.3}$$

The retarded time is again given by $t' = t - |\vec{x}|/c$ such that

$$\omega t' = \omega t - \omega\frac{|\vec{x}|}{c} = \omega t - k|\vec{x}|, \tag{7.9.4}$$

and hence

$$\vec{E}(\vec{x},t) = \frac{I_0 l\omega}{c^2}\sin(\omega t - k|\vec{x}|)\frac{[\vec{e}_z \times \vec{n}] \times \vec{n}}{|\vec{x}|},$$
$$\vec{B}(\vec{x},t) = \frac{I_0 l\omega}{c^2}\sin(\omega t - k|\vec{x}|)\frac{\vec{e}_z \times \vec{n}}{|\vec{x}|}. \tag{7.9.5}$$

Larmor's formula now takes the form

$$P(t) = \frac{2I_0^2 l^2 \omega^2}{3c^3}\sin^2(\omega t - k|\vec{x}|). \tag{7.9.6}$$

Averaging over a period $T = 2\pi/\omega$ we obtain

$$\langle P \rangle = \frac{I_0^2 l^2 \omega^2}{3c^3}. \tag{7.9.7}$$

Chapter 8

Electromagnetic Fields in Matter

Most of the phenomena in our everyday life are consequences of the interaction of electromagnetic fields (photons) with matter (atoms, i.e. electrons and atomic nuclei). Hence, it should be no surprise that this is a non-trivial issue. In fact, the description of electromagnetic interactions at microscopic scales requires the use of quantum mechanics. The quantum mechanics of electromagnetic fields — QED — is part of quantum field theory. This goes far beyond the scope of this course. Here we are most concerned about the macroscopic effects of the interactions of matter and radiation. We will not ask questions about the electromagnetic field inside atoms. Instead we average over a large number of atoms and we ask about the average fields that manifest themselves macroscopically. Since matter consists of charged particles, it can be influenced by electromagnetic fields. For example, the charges may get polarized by an external field, thus creating their own internal field. The total field then is the sum of the two.

8.1 Polarization of Matter

Let us consider a piece of matter consisting of positive and negative charges. When we apply an external electric field it displaces positive and negative charges in opposite directions. This effect is called polarization.[1] Let us consider a

[1] It should be pointed out that this has nothing to do with the polarization of electromagnetic waves.

capacitor first with a vacuum between the plates. On the surface of the metal plates there are surface charge densities ρ_s and $-\rho_s$. The electric field is given by

$$|\vec{E}_e| = \frac{4\pi Q}{A} = 4\pi\rho_s. \tag{8.1.1}$$

We have put an index e to indicate that this is an external electric field (induced by the external charge densities $\pm\rho_s$). The electrostatic potential is given by $\vec{E}_e = -\vec{\nabla}\Phi$ and it takes the form $\Phi(x) = |\vec{E}_e|x$, such that the voltage drop between the plates is

$$V = \Phi(l) - \Phi(0) = |\vec{E}_e|l. \tag{8.1.2}$$

Here l is the distance between the two plates. Hence, the capacity is given by

$$C = \frac{Q}{V} = \frac{A|\vec{E}_e|}{4\pi|\vec{E}_e|l} = \frac{A}{4\pi l}. \tag{8.1.3}$$

Now we take a piece of matter that consists of neutral atoms with positive and negative charges inside. The positive and negative charges may be more or less randomly distributed in the material. When we place the piece of matter inside the capacitor, the external electric field orients the atoms such that their negative end points to the positive plate, while their positive end points to the negative plate. This induces surface charge densities $\mp\rho_{s,p}$ (so-called polarization charges) at the surfaces of the piece of matter (a so-called dielectric). The polarization charges induce an internal electric field $|\vec{E}_i| = 4\pi\rho_{s,p}$ opposing the external electric field. The magnitude of the induced polarization charge densities $\rho_{s,p}$ is a property of the material in question. A medium may be characterized by its polarization vector $\vec{P}$ which measures how far the charges in it are displaced by an external electric field. In particular, the polarization $\vec{P}$ determines the surface polarization charge by

$$\rho_{s,p} = \vec{P}\cdot\vec{n}, \tag{8.1.4}$$

where $\vec{n}$ is a unit-vector normal to the surface. For our capacitor we hence find a total field

$$\vec{E} = \vec{E}_e + \vec{E}_i = \vec{E}_e - 4\pi\rho_{s,p}\vec{e}_x = \vec{E}_e - 4\pi\vec{P}. \tag{8.1.5}$$

This reduces the voltage drop between the plates to

$$V' = |\vec{E}_e - 4\pi\vec{P}|l, \tag{8.1.6}$$

and it increases the capacity of the capacitor with the dielectric material inside to

$$C' = \frac{Q}{V'} = \frac{A|\vec{E}_e|}{4\pi|\vec{E}_e - 4\pi\vec{P}|l} = \frac{A|\vec{E} + 4\pi\vec{P}|}{4\pi|\vec{E}|l} = C\frac{|\vec{E} + 4\pi\vec{P}|}{|\vec{E}|}. \tag{8.1.7}$$

In general, $\vec{P}$ may be a complicated function of the electric field. For small fields, however, one can always linearize this function. The most general linear relation between $\vec{P}$ and $\vec{E}$ is $\vec{P} = M\vec{E}$, where M is some matrix. In fact, there are anisotropic materials (e.g. some crystals) for which $\vec{P}$ and $\vec{E}$ are not parallel. Still, in isotropic materials the matrix M reduces to a single number times the unit matrix. Then one writes

$$4\pi\vec{P} = (\kappa_e - 1)\vec{E}, \tag{8.1.8}$$

where κ_e is the so-called dielectric constant. Then

$$C' = C\kappa_e, \tag{8.1.9}$$

and κ_e tells us by how much the capacity of a capacitor changes if we put a dielectric inside.

In practice, not all of the polarization charge will reside on the surface. In general, we have a volume polarization charge density ρ_p which is related to the polarization by

$$\vec{\nabla} \cdot \vec{P} = -\rho_p. \tag{8.1.10}$$

In addition, there may be "free" charges (like the ones on the surface of the capacitor plates) with a charge density ρ_f. Using the first Maxwell equation we find

$$\vec{\nabla} \cdot \vec{E} = 4\pi(\rho_f + \rho_p) = 4\pi\rho_f - 4\pi\vec{\nabla} \cdot \vec{P}. \tag{8.1.11}$$

It is common to introduce the quantity

$$\vec{D} = \vec{E} + 4\pi\vec{P}, \tag{8.1.12}$$

such that

$$\vec{\nabla} \cdot \vec{D} = 4\pi\rho_f, \tag{8.1.13}$$

which looks similar to the first Maxwell equation again.

The homogeneous Maxwell equations without charge and current densities on the right-hand side

$$\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t\vec{B} = 0, \ \vec{\nabla} \cdot \vec{B} = 0, \tag{8.1.14}$$

need not be rewritten in the presence of matter.

8.2 Magnetization

The fourth Maxwell equation

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}\vec{j}, \tag{8.2.1}$$

however, should be rewritten in the presence of matter. First of all, we want to separate the effects of polarization charges from the ones of the free charges. When the polarization changes with time, the motion of the polarization charges generates a current

$$\vec{j}_p = \partial_t \vec{P}, \tag{8.2.2}$$

which indeed obeys a continuity equation

$$\partial_t \rho_p + \vec{\nabla} \cdot \vec{j}_p = -\partial_t \vec{\nabla} \cdot \vec{P} + \vec{\nabla} \cdot \partial_t \vec{P} = 0. \tag{8.2.3}$$

Besides that we have the current of the free charges $\vec{j}_f$ as well as internal currents $\vec{j}_m$ due to a possible magnetization of the material, such that

$$\vec{\nabla} \times \vec{B} - \frac{1}{c}\partial_t \vec{E} = \frac{4\pi}{c}(\partial_t \vec{P} + \vec{j}_f + \vec{j}_m). \tag{8.2.4}$$

The current $\vec{j}_m$ is related to the magnetization $\vec{M}$ of the material by

$$\vec{j}_m = c\vec{\nabla} \times \vec{M}. \tag{8.2.5}$$

Writing

$$\vec{H} = \vec{B} - 4\pi\vec{M}, \tag{8.2.6}$$

we then obtain

$$\vec{\nabla} \times \vec{H} - \frac{1}{c}\partial_t \vec{D} = \frac{4\pi}{c}\vec{j}_f. \tag{8.2.7}$$

Altogether, Maxwell's equations in matter take the form

$$\begin{aligned}
&\vec{\nabla} \cdot (\vec{E} + 4\pi\vec{P}) = 4\pi\rho_f \;\Rightarrow\; \vec{\nabla} \cdot \vec{D} = 4\pi\rho_f \\
&\vec{\nabla} \times \vec{E} + \frac{1}{c}\partial_t \vec{B} = 0, \\
&\vec{\nabla} \cdot \vec{B} = 0, \\
&\vec{\nabla} \times (\vec{B} - 4\pi\vec{M}) - \frac{1}{c}\partial_t(\vec{E} + 4\pi\vec{P}) = \frac{4\pi}{c}\vec{j}_f \;\Rightarrow \\
&\vec{\nabla} \times \vec{H} - \frac{1}{c}\partial_t \vec{D} = \frac{4\pi}{c}\vec{j}_f.
\end{aligned} \tag{8.2.8}$$

In general, the polarization and the magnetization are themselves functions of $\vec{E}$ and $\vec{B}$, i.e. $\vec{P} = \vec{P}(\vec{E})$, $\vec{M} = \vec{M}(\vec{B})$, and the specific form of the function depends on the material in question. Relatively simple so-called linear materials obey

$$4\pi\vec{P} = (\kappa_e - 1)\vec{E},\ 4\pi\vec{M} = (1 - \frac{1}{\kappa_m})\vec{B}, \tag{8.2.9}$$

where κ_m is the so-called magnetic permeability. For linear materials we have

$$\vec{E} + 4\pi\vec{P} = \vec{E} + (\kappa_e - 1)\vec{E} = \kappa_e\vec{E},\ \vec{B} - 4\pi\vec{M} = \vec{B} - (1 - \frac{1}{\kappa_m})\vec{B} = \frac{\vec{B}}{\kappa_m}. \tag{8.2.10}$$

When the material is a conductor one also has

$$\vec{j}_f = \sigma\vec{E}, \tag{8.2.11}$$

where σ is the conductivity. In this case, Maxwell's equations take the form

$$\vec{\nabla}\cdot(\kappa_e\vec{E}) = 4\pi\rho_f,\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\partial_t\vec{B} = 0,$$
$$\vec{\nabla}\cdot\vec{B} = 0,\ \vec{\nabla}\cdot(\frac{\vec{B}}{\kappa_m}) - \frac{1}{c}\partial_t(\kappa_e\vec{E}) = \frac{4\pi\sigma}{c}\vec{E}. \tag{8.2.12}$$

These equations are again linear in $\vec{E}$ and $\vec{B}$. This is the reason why materials that obey them are called linear. One should know that there are also non-linear materials. In this course we will only study linear materials.

8.3 Plane Waves in a Medium

Let us consider wave propagation in a linear medium. We expect that plane wave solutions still exist, so we are led to the following ansatz for a wave propagating in the z-direction

$$\vec{E}(z,t) = E_0\exp(i(kz - \omega t))\vec{e}_x,\ \vec{B}(z,t) = B_0\exp(i(kz - \omega t))\vec{e}_y. \tag{8.3.1}$$

Now we insert this ansatz in the Maxwell equation for a linear medium and obtain

$$\vec{\nabla}\cdot\vec{E} = 0,\ \vec{\nabla}\cdot\vec{B} = 0. \tag{8.3.2}$$

We conclude that in order to be consistent ρ_f must vanish in this case. Further, we have

$$\begin{aligned}
&\vec{\nabla} \times \vec{E} = \partial_z E_x \vec{e}_y = ikE_0 \exp(i(kz - \omega t))\vec{e}_y, \\
&\partial_t \vec{B} = -i\omega B_0 \exp(i(kz - \omega t))\vec{e}_y \ \Rightarrow \ kE_0 = \omega B_0 \ \Rightarrow \ \frac{B_0}{E_0} = \frac{k}{\omega}, \\
&\vec{\nabla} \times \vec{B} = -\partial_z B_y \vec{e}_x = -ikB_0 \exp(i(kz - \omega t))\vec{e}_x, \\
&\partial_t \vec{E} = -i\omega E_0 \exp(i(kz - \omega t))\vec{e}_x \ \Rightarrow \\
&-ik\frac{B_0}{\kappa_m} = \frac{1}{c}\kappa_e(-i\omega E_0) + \frac{4\pi}{c}\sigma E_0 \ \Rightarrow \\
&\frac{B_0}{E_0}k = \frac{1}{c}\kappa_e\kappa_m\omega + i\frac{4\pi}{c}\sigma\kappa_m \ \Rightarrow \ k^2 = \omega^2\frac{1}{c^2}\kappa_e\kappa_m + i\omega\frac{4\pi}{c}\sigma\kappa_m. \qquad (8.3.3)
\end{aligned}$$

We see that in general k will be complex. The imaginary part of k corresponds to damping of the electromagnetic wave in matter. In fact, for some materials even κ_e, κ_m, and σ may be complex. Let us make our life simpler by considering a perfect dielectric, i.e. κ_e and κ_m are real and $\sigma = 0$. Then we find the phase velocity

$$v = \frac{\omega}{k} = \frac{c}{\sqrt{\kappa_e\kappa_m}}. \qquad (8.3.4)$$

In a vacuum we have $\kappa_e = \kappa_m = 1$, such that we then recover $v = c$. The index of refraction of the medium is defined as

$$n = \frac{c}{v} = \sqrt{\kappa_e\kappa_m}. \qquad (8.3.5)$$

Appendix A

Physical Units

Physical units are to a large extent a convention influenced by the historical development of physics. In electrodynamics we encounter a variety of units. In particular, it seems that most pioneers of electrodynamics including Ampere, Coulomb, Gauss, Ohm, Tesla, Volta, and several others have been honored by naming a physical unit after them. All these units are man-made conventions. In order to understand the physics literature, we must familiarize ourselves with various choices of units, even if the conventions may seem unnatural from today's point of view. Interestingly, there are also natural units which express physical quantities in terms of fundamental constants of Nature: Newton's gravitational constant G, the velocity of light c, and Planck's quantum h. In this chapter, we consider the issue of physical units from a general point of view.

A.1 Units of Time

Time is measured by counting periodic phenomena. The most common periodic phenomenon in our everyday life is the day and, related to that, the year. Hence, it is no surprise that the first precise clock used by humans was the solar system. In our life span, if we stay healthy, we circle around the sun about 80 times, every circle defining one year. During one year, we turn around the earth's axis 365 times, every turn defining one day. When we build a pendulum clock, it helps us to divide the day into $24 \times 60 \times 60 = 86400$ seconds. The second (about the duration of one heart beat) is perhaps the shortest time interval that people care about in their everyday life. However, as physicists we do not stop there, because we need to be able to measure much shorter time intervals, in particular, as we investigate physics of fundamental objects such as atoms or individual elementary particles. Instead of using the solar system as a gigantic mechanical clock, the most accurate modern clock is a tiny quantum mechanical analog of the solar system — an individual cesium atom. Instead of defining 1 sec as one turn around earth's axis divided into 86400 parts, the modern definition of 1 sec corresponds to 9192631770 periods of a particular microwave transition of the cesium atom. The atomic cesium clock is very accurate and defines a reproducible standard of time. Unlike the solar system, this standard could be established anywhere in the Universe. Cesium atoms are fundamental objects which work in the same way everywhere at all times.

A.2 Units of Length

The lengths we care about most in our everyday life are of the order of the size of our body. It is therefore not surprising that, in order to define a standard, some stick — defined to be 1 meter — was deposited near Paris a long time ago. Obviously, this is a completely arbitrary man-made unit. A physicist elsewhere in the Universe would not want to subscribe to that convention. A trip to Paris just to measure a length would be too inconvenient. How can we define a natural standard of length that would be easy to establish anywhere at all times? For example, one could say that the size of our cesium atom sets such a standard. Still, this is not how this is handled. Einstein has taught us that the velocity of light c in vacuum is an absolute constant of Nature, independent of any observer. Instead of referring to the stick in Paris, one now defines the meter through c and the second as

$$c = 2.99792456 \times 10^8 \text{m sec}^{-1} \;\Rightarrow\; 1\text{m} = 3.333564097 \times 10^{-7} c \text{ sec}. \qquad \text{(A.2.1)}$$

In other words, the measurement of a distance is reduced to the measurement of the time it takes a light signal to pass that distance. Since relativity theory tells us that light travels with the same speed everywhere at all times, we have thus established a standard of length that could easily be used by physicists anywhere in the Universe.

A.3 Units of Mass

Together with the meter stick, a certain amount of platinum-iridium alloy was deposited near Paris a long time ago. The corresponding mass was defined to be one kilogram. Obviously, this definition is as arbitrary as that of the meter. Since the original kilogram has been moved around too often over the past 100 years or so, it has lost some weight and no longer satisfies modern requirements for a standard of mass. One might think that it would be best to declare, for example, the mass of a single cesium atom as an easily reproducible standard of mass. While this is true in principle, it is inconvenient in practical experimental situations. Accurately measuring the mass of a single atom is highly non-trivial. Instead, it was decided to produce a more stable kilogram that will remain constant for the next 100 years or so. Maintaining a standard is important business for experimentalists, but a theorist doesn't care much about the arbitrarily chosen amount of matter deposited near Paris.

A.4 Natural Planck Units

Irrespective of practical considerations, it is interesting to think about standards that are natural from a theoretical point of view. There are three fundamental constants of Nature that can help us in this respect. First, to establish a standard of length, we have already used the velocity of light c which plays a central role in the theory of relativity. Quantum mechanics provides us with another fundamental constant — Planck's quantum (divided by 2π)

$$\hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34} \text{kg m}^2 \text{sec}^{-1}. \quad \text{(A.4.1)}$$

As theorists, we are not terribly excited about knowing the value of $\hbar$ in units of kilograms, meters, and seconds, because these are arbitrarily chosen man-made units. Instead, it would be natural to use $\hbar$ itself as a basic unit of dimension energy times time. A third dimensionful fundamental constant that suggests itself through general relativity is Newton's gravitational constant

$$G = 6.6720 \times 10^{-11} \text{kg}^{-1} \text{m}^3 \text{sec}^{-2}. \quad \text{(A.4.2)}$$

Using c, $\hbar$, and G we can define natural units also known as Planck units. First there are the Planck time

$$t_{\text{Planck}} = \sqrt{\frac{G\hbar}{c^5}} = 5.3904 \times 10^{-44} \text{sec}, \quad \text{(A.4.3)}$$

and the Planck length

$$l_{\text{Planck}} = \sqrt{\frac{G\hbar}{c^3}} = 1.6160 \times 10^{-35} \text{m}, \quad \text{(A.4.4)}$$

which represent the shortest times and distances relevant in physics. Today we are very far from exploring such short length- and time-scales experimentally. It is even believed that our classical concepts of space and time will break down at the Planck scale. One may speculate that at the Planck scale space and time become discrete, and that l_{Planck} and t_{Planck} may represent the shortest elementary quantized units of space and time. We can also define the Planck mass

$$M_{\text{Planck}} = \sqrt{\frac{\hbar c}{G}} = 2.1768 \times 10^{-8}\text{kg}, \tag{A.4.5}$$

which is the highest mass scale relevant to elementary particle physics.

Planck units would not be very practical in our everyday life. For example, a lecture would last about $10^{48} t_{\text{Planck}}$, the distance from the lecture hall to the cafeteria would be about $10^{37} l_{\text{Planck}}$, and our lunch would weigh about $10^{7} M_{\text{Planck}}$. Still, these are the natural units that Nature suggests to us and it is interesting to ask why we exist at scales so far removed from the Planck scale. For example, we may ask why the Planck mass corresponds to about 10^{-8} kg. In some sense this is just a historical question. The amount of matter deposited near Paris to define the kilogram obviously was an arbitrary man-made unit. However, if we assume that the kilogram was chosen because it is a reasonable fraction of our own weight, it may be phrased as a biological question: Why do intelligent creatures weigh about $10^{10} M_{\text{Planck}}$? We can turn the question into a physics problem when we think about the physical origin of mass. Indeed (up to tiny corrections) the mass of the matter that surrounds us is contained in atomic nuclei which consist of protons and neutrons with masses

$$\begin{aligned} M_p &= 1.67266 \times 10^{-27}\text{kg} = 7.6840 \times 10^{-20} M_{\text{Planck}}, \\ M_n &= 1.67496 \times 10^{-27}\text{kg} = 7.6946 \times 10^{-20} M_{\text{Planck}}. \end{aligned} \tag{A.4.6}$$

Why are protons and neutrons so light compared to the Planck mass? This physics question has actually been understood at least qualitatively using the property of asymptotic freedom of quantum chromodynamics (QCD) — the quantum field theory of quarks and gluons whose interaction energy explains the masses of protons and neutrons. The Nobel prize of the year 2004 was awarded to David Gross, David Politzer, and Frank Wilczek for the understanding of asymptotic freedom. As explained in the introduction, eq.(A.4.6) also explains why gravity is an extremely weak force.

The strength of electromagnetic interactions is determined by the quantized charge unit e (the electric charge of a proton). In natural Planck units it gives rise to the experimentally determined fine-structure constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036}. \tag{A.4.7}$$

The strength of electromagnetism is determined by this pure number which is completely independent of any man-made conventions. It is a very interesting physics question to ask why α has this particular value. At the moment, physicists have no clue how to answer this question. These days it is popular to refer to the anthropic principle. If α would be different, all of atomic physics and thus all of chemistry would work differently, and life as we know it might be impossible. According to the anthropic principle, we can only live in a part of a Multiverse (namely in our Universe) with a "life-friendly" value of α. The author does not subscribe to this way of thinking. Since we will always be confined to our Universe, we cannot falsify the anthropic argument. It thus does not belong to rigorous scientific thinking. This does not mean that one should not think that way, but it is everybody's private business. The author is optimistic that some day some smart theoretical physicist will understand why α takes the above experimentally observed value.

A.5 Units of Charge

As we have seen, in Planck units the strength of electromagnetism is given by the fine-structure constant which is a dimensionless number independent of any man-made conventions. Obviously, the natural unit of charge that Nature suggests to us is the elementary charge quantum e — the electric charge of a single proton. Of course, in experiments on macroscopic scales one usually deals with an enormous number of elementary charges at the same time. Just like using Planck units in everyday life is not very practical, measuring charge in units of

$$e = \sqrt{\frac{\hbar c}{137.036}} = 1.5189 \times 10^{-14} \mathrm{kg}^{1/2} \mathrm{m}^{3/2} \mathrm{sec}^{-1} \tag{A.5.1}$$

can also be inconvenient. For this purpose large amounts of charge have also been used to define charge units. For example, one electrostatic unit is defined as

$$1\mathrm{esu} = 2.0819 \times 10^9 e = 3.1622 \times 10^{-5} \mathrm{kg}^{1/2} \mathrm{m}^{3/2} \mathrm{sec}^{-1}. \tag{A.5.2}$$

This charge definition has the curious property that the Coulomb force between two electrostatic charge units at a distance of 1 centimeter is

$$1 \frac{\mathrm{esu}^2}{\mathrm{cm}^2} = 10^{-5} \mathrm{kg\ m\ sec}^{-2} = 10^{-5} \mathrm{N} = 1\mathrm{dyn}. \tag{A.5.3}$$

Again, the origin of this charge unit is at best of historical interest and just represents another man-made convention.

A.6 Dimensions of Electromagnetic Quantities

Let us consider the dimension of various quantities relevant to electromagnetism. Using the force on a particle of charge q

$$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}), \tag{A.6.1}$$

and the fact that force has the dimension

$$d_F = \text{mass} \times \text{length} \times \text{time}^{-2}, \tag{A.6.2}$$

and charge has dimension

$$d_q = \text{mass}^{1/2} \times \text{length}^{3/2} \times \text{time}^{-1}, \tag{A.6.3}$$

we conclude that the dimension of the electric and magnetic fields is

$$d_E = d_B = \frac{d_F}{d_q} = \text{mass}^{1/2} \times \text{length}^{-1/2} \times \text{time}^{-1}. \tag{A.6.4}$$

The scalar and vector potentials are related to static electric and magnetic fields by

$$\vec{E} = -\vec{\nabla}\Phi, \ \vec{B} = \vec{\nabla} \times \vec{A}, \tag{A.6.5}$$

which implies

$$d_\Phi = d_E \times \text{length} = d_A = d_B \times \text{length} = \text{mass}^{1/2} \times \text{length}^{1/2} \times \text{time}^{-1}. \tag{A.6.6}$$

The dimension of the charge density is

$$d_\rho = d_q \times \text{length}^{-3} = \text{mass}^{1/2} \times \text{length}^{-3/2} \times \text{time}^{-1}, \tag{A.6.7}$$

and the one of the current density is

$$d_j = d_q \times \text{length}^{-2} \times \text{time}^{-1} = \text{mass}^{1/2} \times \text{length}^{-1/2} \times \text{time}^{-2}. \tag{A.6.8}$$

A.7 From Gaussian to MKS Units

In this course, we will use the so-called Gaussian system in which charge is of dimension mass$^{1/2}\times$length$^{3/2}\times$time^{-1}, and the velocity of light plays a prominent role. This is the system that is natural from a theoretical point of view. An alternative used in many technical applications and described in several textbooks is the MKS system. Although it is natural and completely sufficient to assign the above dimension to charge, in the MKS system charge is measured in a new independent unit called a Coulomb in honor of Charles-Augustin de Coulomb (1736 - 1806). In MKS units Coulomb's law for the force between two charges q_1 and q_2 at a distance r takes the form

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}. \qquad \text{(A.7.1)}$$

The only purpose of the quantity ε_0 is to compensate the dimension of the charges. For example, in MKS units the elementary charge quantum corresponds to e' with

$$e = \frac{e'}{\sqrt{4\pi\varepsilon_0}}. \qquad \text{(A.7.2)}$$

The Coulomb is then defined as

$$1\text{Cb} = 6.2414 \times 10^{18} e'. \qquad \text{(A.7.3)}$$

Comparing with eq.(A.5.2) one obtains

$$1\text{Cb} = 10c\sqrt{4\pi\varepsilon_0}\ \text{sec m}^{-1}\text{esu}. \qquad \text{(A.7.4)}$$

In order to compare with textbooks using the MKS system, table 1 allows us to translate between the two systems. In the MKS system the force on a charge q is given by

$$\vec{F} = q(\vec{E} + \vec{v}\times\vec{B}) \qquad \text{(A.7.5)}$$

and Maxwell's equations take the form

$$\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0},\ \vec{\nabla}\times\vec{E} + \partial_t\vec{B} = 0,$$
$$\vec{\nabla}\cdot\vec{B} = 0,\ \vec{\nabla}\times\vec{B} - \varepsilon_0\mu_0\partial_t\vec{E} = \mu_0\vec{j}. \qquad \text{(A.7.6)}$$

For example, the last equation results by turning the equation

$$\vec{\nabla}\times\vec{B} - \frac{1}{c}\partial_t\vec{E} = \frac{4\pi}{c}\vec{j} \qquad \text{(A.7.7)}$$

in the Gaussian system into

$$\vec{\nabla}\times\sqrt{\frac{4\pi}{\mu_0}}\vec{B} - \sqrt{\varepsilon_0\mu_0}\partial_t\sqrt{4\pi\varepsilon_0}\vec{E} = 4\pi\sqrt{\varepsilon_0\mu_0}\frac{1}{\sqrt{4\pi\varepsilon_0}}\vec{j}. \qquad \text{(A.7.8)}$$

Quantity	Gaussian	MKS
Velocity of light	c	$1/\sqrt{\varepsilon_0\mu_0}$
Charge density	ρ	$\rho/\sqrt{4\pi\varepsilon_0}$
Current density	$\vec{j}$	$\vec{j}/\sqrt{4\pi\varepsilon_0}$
Electric field	$\vec{E}$	$\sqrt{4\pi\varepsilon_0}\vec{E}$
Scalar potential	Φ	$\sqrt{4\pi\varepsilon_0}\Phi$
Magnetic field	$\vec{B}$	$\sqrt{4\pi/\mu_0}\vec{B}$
Vector potential	$\vec{A}$	$\sqrt{4\pi/\mu_0}\vec{A}$
Energy density	$u = \frac{1}{8\pi}(\vec{E}^2 + \vec{B}^2)$	$u = \frac{1}{2}(\varepsilon_0\vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2)$
Poynting vector	$\vec{S} = \frac{c}{4\pi}\vec{E} \times \vec{B}$	$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$

Table A.1: *Translation table between the Gaussian and the MKS system.*

A.8 Various Units in the MKS system

To honor Allessandro Volta (1845 - 1897) the unit for potential difference is called a Volt and is defined as

$$1\text{V} = 1\text{kg m}^2\ \text{sec}^{-2}\text{Cb}^{-1}. \tag{A.8.1}$$

To maintain the Volt standard one uses the Josephson effect related to the current flowing between two superconductors separated by a thin insulating layer — a so-called Josephson junction. The effect was predicted by Brian David Josephson in 1962, for which he received the Nobel prize in 1973. A Josephson junction provides a very accurate standard for potential differences in units of the fundamental magnetic flux quantum $h/2e$.

In honor of Andre-Marie Ampere (1775 - 1836), the unit of current is called an Ampere which is defined as

$$1\text{A} = 1\text{Cb sec}^{-1}. \tag{A.8.2}$$

Using Ohm's law, the standard of the Ampere is maintained using the standards of potential difference and resistance.

The unit of resistance is one Ohm in honor of Georg Simon Ohm (1789 - 1854) who discovered that the current I flowing through a conductor is proportional to the applied voltage U. The proportionality constant is the resistance R and Ohm's law thus reads $U = RI$. Historically, the Ohm was defined at an international conference on electricity in 1881 as

$$1\Omega = 1\text{V A}^{-1} = 1\text{kg m}^2\ \text{sec}^{-1}\text{Cb}^{-2}. \tag{A.8.3}$$

Once this was decided, the question arose how to maintain the standard of resistance. In other words, which conductor has a resistance of 1 Ohm? At another international conference in 1908 people agreed that a good realization of a conductor with a resistance of one Ohm are 14.4521 g of mercury in a column of length 1.063 m with constant cross-section at the temperature of melting ice. However, later it was realized that this was not completely consistent with the original definition of the Ohm. Today the standard of the Ohm is maintained extremely accurately using an interesting quantum phenomenon — the quantum Hall effect — discovered by Klaus von Klitzing in 1980, for which he received the Nobel prize in 1985. Von Klitzing observed that at extremely low temperatures the resistance of a planar conductor in a strong perpendicular magnetic field is quantized in units of h/e^2, where h is Planck's quantum and e is the basic unit of electric charge. The quantum Hall effect provides us with an accurate and reproducible measurement of resistance and is now used to maintain the standard of the Ohm by using the fact that

$$1\Omega = 3.8740459 \times 10^{-5} \frac{h}{e^2}. \qquad \text{(A.8.4)}$$

In honor of Nikola Tesla (1856 - 1943), magnetic fields are measured in Tesla defined as

$$1\text{T} = 1\text{kg sec}^{-1}\text{Cb}^{-1}. \qquad \text{(A.8.5)}$$

The corresponding unit in the Gaussian system is 1 Gauss, honoring Carl Friedrich Gauss (1777 - 1855), with 10^4 Gauss corresponding to 1 Tesla.

The units Faraday and Maxwell in honor of the great pioneers of electromagnetism Michael Faraday (1791 - 1867) and James Clerk Maxwell (1831 - 1879) are no longer in use.

Appendix B

Vector Analysis and Integration Theorems

Before we can study electrodynamics we need to equip ourselves with some necessary mathematical tools such as vector analysis and integration theorems. First, we'll remind ourselves of the concepts of scalars and vectors and then we'll apply these concepts to construct scalar and vector fields as well as their derivatives. Finally, we'll introduce Gauss' and Stokes' theorems.

B.1 Scalars and Vectors

Physical phenomena take place in space. Hence, their mathematical description naturally involves space points

$$\vec{x} = (x_1, x_2, x_3). \tag{B.1.1}$$

Here x_1, x_2, and x_3 are just the Euclidean x-, y-, and z-components of the 3-dimensional vector $\vec{x}$. The fact that $\vec{x}$ is a vector means that it behaves in a particular way under spatial rotations. In particular, in a rotated coordinate system, the components x_1, x_2, and x_3 change accordingly. Let us consider a real 3×3 orthogonal rotation matrix O with determinant 1. For an orthogonal matrix

$$O^T O = O O^T = \mathbb{1}, \tag{B.1.2}$$

where T denotes transpose. Under such a spatial rotation the vector $\vec{x}$ transforms into

$$\vec{x}\,' = O\vec{x}. \tag{B.1.3}$$

The scalar product of two vectors

$$\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + x_3 y_3 \tag{B.1.4}$$

is unaffected by the spatial rotation, i.e.

$$\vec{x}\,' \cdot \vec{y}\,' = \vec{x} O^T O \vec{y} = \vec{x} \mathbb{1} \vec{y} = \vec{x} \cdot \vec{y}. \tag{B.1.5}$$

Quantities that don't change under spatial rotations are known as scalars (hence the name "scalar" product). The length of a vector is the square root of its scalar product with itself, i.e.

$$|\vec{x}| = \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{x_1^2 + x_2^2 + x_3^2}. \tag{B.1.6}$$

Obviously, the length of a vector also is a scalar, i.e. it does not change under spatial rotations.

Two 3-dimensional vectors $\vec{x}$ and $\vec{y}$ can be combined to another vector $\vec{z}$ by forming the vector cross product

$$\vec{z} = \vec{x} \times \vec{y} = (x_1 y_2 - x_2 y_1, x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3). \tag{B.1.7}$$

It may not be obvious that $\vec{z}$ is indeed a vector. Obviously, it has three components, but do they transform appropriately under spatial rotations? Indeed, one can show that

$$\vec{z}\,' = \vec{x}\,' \times \vec{y}\,' = O\vec{x} \times O\vec{y} = O\vec{z}. \tag{B.1.8}$$

B.2 Fields

The concept of fields is central to electrodynamics and all other field theories including general relativity and the standard model of elementary particle physics. A field describes physical degrees of freedom attached to points $\vec{x}$ in space. The simplest example is a real-valued scalar field $\Phi(\vec{x})$. Here $\Phi(\vec{x})$ is a real number attached to each point $\vec{x}$ in space. In other words, $\Phi(\vec{x})$ is a real-valued function of the three spatial coordinates. In electrodynamics, fields in general depend on both space and time. For the moment we limit ourselves to static (i.e. time-independent) fields. When we say that $\Phi(\vec{x})$ is a scalar field, we mean that its value is unaffected by a spatial rotation O, i.e.

$$\Phi(\vec{x}\,')' = \Phi(\vec{x}) = \Phi(O^T \vec{x}\,'). \tag{B.2.1}$$

Of course, the spatial point $\vec{x}$ itself is rotated to the new position $\vec{x}\,' = O\vec{x}$, such that $\vec{x} = O^T \vec{x}\,'$. A physical example for a scalar field is the scalar potential in electrodynamics.

Another important field in electrodynamics is the electric field

$$\vec{E}(\vec{x}) = (E_1(\vec{x}), E_2(\vec{x}), E_3(\vec{x})). \tag{B.2.2}$$

The electric field is a vector field, i.e. there is not just a single number attached to each point in space. Instead the field $\vec{E}(\vec{x})$ is a collection of three real numbers $E_1(\vec{x})$, $E_2(\vec{x})$, and $E_3(\vec{x})$. Again, these three components transform in a specific way under spatial rotations

$$\vec{E}(\vec{x}\,')' = O\vec{E}(\vec{x}) = O\vec{E}(O^T \vec{x}\,'). \tag{B.2.3}$$

The magnetic field $\vec{B}(\vec{x})$ is another important vector field in electrodynamics. Besides that we'll encounter yet another vector field $\vec{A}(\vec{x})$, the so-called vector potential. It is easy to show that the product of a scalar and a vector field $\Phi(\vec{x})\vec{A}(\vec{x})$ is again a vector field, while the scalar product of two vector fields $\vec{A}(\vec{x}) \cdot \vec{B}(\vec{x})$ is a scalar field. Similarly, the vector cross product of two vector fields $\vec{A}(\vec{x}) \times \vec{B}(\vec{x})$ is again a vector field.

B.3 Differential Operators

Since fields are functions of space, it is natural to take spatial derivatives of them. A most important observation is that the derivatives with respect to the various Euclidean coordinates form a vector — the so-called "Nabla" operator

$$\vec{\nabla} = (\partial_1, \partial_2, \partial_3) = \left(\frac{d}{dx_1}, \frac{d}{dx_2}, \frac{d}{dx_3}\right). \tag{B.3.1}$$

When we say that $\vec{\nabla}$ is a vector we again mean that it transforms appropriately under spatial rotations, i.e.

$$\vec{\nabla}\,' = (\frac{d}{dx_1'}, \frac{d}{dx_2'}, \frac{d}{dx_3'}) = O\vec{\nabla}. \tag{B.3.2}$$

This immediately follows from the chain rule relation

$$\frac{d}{dx'_i} = \sum_j \frac{\partial x_j}{\partial x'_i}\frac{d}{dx_j} = \sum_j O^T_{ji}\frac{d}{dx_j} = \sum_j O_{ij}\frac{d}{dx_j}. \tag{B.3.3}$$

The operator $\vec{\nabla}$ acts on a scalar field as a gradient

$$\vec{\nabla}\Phi(\vec{x}) = (\partial_1\Phi(\vec{x}), \partial_2\Phi(\vec{x}), \partial_3\Phi(\vec{x})) = \left(\frac{d\Phi(\vec{x})}{dx_1}, \frac{d\Phi(\vec{x})}{dx_2}, \frac{d\Phi(\vec{x})}{dx_3}\right). \tag{B.3.4}$$

In this way the scalar field $\Phi(\vec{x})$ is turned into the vector field $\vec{\nabla}\Phi(\vec{x})$. The operator $\vec{\nabla}$ can also act on a vector field. First, one can form the scalar product

$$\vec{\nabla}\cdot\vec{A}(\vec{x}) = \partial_1 A_1(\vec{x}) + \partial_2 A_2(\vec{x}) + \partial_3 A_3(\vec{x}) = \frac{dA_1(\vec{x})}{dx_1} + \frac{dA_2(\vec{x})}{dx_2} + \frac{dA_3(\vec{x})}{dx_3}. \tag{B.3.5}$$

Then the operator $\vec{\nabla}$ acts as a divergence and turns the vector field $\vec{A}(\vec{x})$ into the scalar field $\vec{\nabla}\cdot\vec{A}(\vec{x})$. One can also form the vector cross product

$$\begin{aligned}\vec{\nabla}\times\vec{A}(\vec{x}) &= (\partial_1 A_2(\vec{x}) - \partial_2 A_1(\vec{x}), \partial_2 A_3(\vec{x}) - \partial_3 A_2(\vec{x}), \partial_3 A_1(\vec{x}) - \partial_1 A_3(\vec{x})) \\ &= \left(\frac{dA_2(\vec{x})}{dx_1} - \frac{dA_1(\vec{x})}{dx_2}, \frac{dA_3(\vec{x})}{dx_2} - \frac{dA_2(\vec{x})}{dx_3}, \frac{dA_1(\vec{x})}{dx_3} - \frac{dA_3(\vec{x})}{dx_1}\right).\end{aligned} \tag{B.3.6}$$

In this case, $\vec{\nabla}$ acts as a curl and turns the vector field $\vec{A}(\vec{x})$ into the vector field $\vec{\nabla}\times\vec{A}(\vec{x})$. All these operations are relevant in electrodynamics. For example, static electric and magnetic fields are related to the scalar and vector potentials by

$$\vec{E}(\vec{x}) = -\vec{\nabla}\Phi(\vec{x}),\ \vec{B}(\vec{x}) = \vec{\nabla}\times\vec{A}(\vec{x}). \tag{B.3.7}$$

It is easy to show that

$$\vec{\nabla}\times\vec{\nabla}\Phi(\vec{x}) = 0,\ \vec{\nabla}\cdot\vec{\nabla}\times\vec{A}(\vec{x}) = 0. \tag{B.3.8}$$

Still, one can form a non-zero second derivative

$$\Delta = \vec{\nabla}\cdot\vec{\nabla} = \partial_1^2 + \partial_2^2 + \partial_3^2 = \frac{d^2}{dx_1^2} + \frac{d^2}{dx_2^2} + \frac{d^2}{dx_3^2}, \tag{B.3.9}$$

which is known as the Laplacian.

B.4 Integration Theorems

We are all familiar with the simple theorem for the 1-dimensional integration of the derivative of a function $f(x)$

$$\int_a^b dx\ \frac{df(x)}{dx} = f(a) - f(b). \tag{B.4.1}$$

Here the points a and b define the boundary of the integration region (the interval $[a, b]$). This simple theorem has generalizations in higher dimensions associated with the names of Gauss and Stokes. Here we focus on three dimensions. Instead of proving Gauss' and Stokes' theorems we simply state them and refer to the mathematical literature for proofs. Approximating differentiation by finite differences and integration by Riemann sums, it is straightforward to prove both theorems.

Gauss' integration theorem deals with the volume integral of the divergence of some vector field $\vec{A}(\vec{x})$

$$\int_V d^3x\ \vec{\nabla} \cdot \vec{A}(\vec{x}) = \int_{\partial V} d^2\vec{f} \cdot \vec{A}(\vec{x}). \tag{B.4.2}$$

Here V is the 3-dimensional integration volume and ∂V is its boundary, a closed 2-dimensional surface. The unit vector normal to the surface is $\vec{f}$.

Similarly, Stokes' theorem is concerned with the surface integral of the curl of a vector field

$$\int_S d^2\vec{f} \cdot \vec{\nabla} \times \vec{A}(\vec{x}) = \int_{\partial S} d\vec{l} \cdot \vec{A}(\vec{x}). \tag{B.4.3}$$

Here S is the 2-dimensional surface to be integrated over and ∂S is its boundary, a closed curve. The unit vector $\vec{l}$ is tangential to this curve.

Appendix C

Operators in Curvilinear Coordinates

Besides ordinary Cartesian coordinates, spherical as well as cylindrical coordinates play an important role. They can make life a lot easier in situations which are spherically or cylindrically symmetric.

C.1 Operators in Spherical Coordinates

Spherical coordinates are defined as

$$\begin{aligned}
\vec{x} &= r(\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = r\vec{e}_r, \\
\vec{e}_r &= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \\
\vec{e}_\theta &= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta), \\
\vec{e}_\varphi &= (-\sin\varphi, \cos\varphi, 0).
\end{aligned} \tag{C.1.1}$$

In spherical coordinates the gradient of a scalar field $\Phi(r, \theta, \varphi)$ takes the form

$$\vec{\nabla}\Phi = \partial_r\Phi\ \vec{e}_r + \frac{1}{r}\partial_\theta\Phi\ \vec{e}_\theta + \frac{1}{r\sin\theta}\partial_\varphi\Phi\ \vec{e}_\varphi, \tag{C.1.2}$$

and the Laplacian is given by

$$\Delta\Phi = \frac{1}{r^2}\partial_r(r^2\partial_r\Phi) + \frac{1}{r^2\sin\theta}\partial_\theta(\sin\theta\partial_\theta\Phi) + \frac{1}{r^2\sin^2\theta}\partial_\varphi^2\Phi. \tag{C.1.3}$$

A general vector field can be written as

$$\vec{E}(\vec{x}) = E_r(r,\theta,\varphi)\vec{e}_r + E_\theta(r,\theta,\varphi)\vec{e}_\theta + E_\varphi(r,\theta,\varphi)\vec{e}_\varphi. \tag{C.1.4}$$

The general expression for the divergence of a vector field in spherical coordinates takes the form

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\partial_r(r^2 E_r) + \frac{1}{r\sin\theta}\partial_\theta(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\partial_\varphi E_\varphi, \tag{C.1.5}$$

while the curl is given by

$$\begin{aligned}\vec{\nabla}\times\vec{E} &= \left[\frac{1}{r\sin\theta}\partial_\theta(\sin\theta E_\varphi) - \frac{1}{r\sin\theta}\partial_\varphi E_\theta\right]\vec{e}_r \\ &+ \left[\frac{1}{r\sin\theta}\partial_\varphi E_r - \frac{1}{r}\partial_r(rE_\varphi)\right]\vec{e}_\theta + \left[\frac{1}{r}\partial_r(rE_\theta) - \frac{1}{r}\partial_\theta E_r\right]\vec{e}_\varphi.\end{aligned} \tag{C.1.6}$$

C.2 Operators in Cylindrical Coordinates

Cylindrical coordinates are defined as

$$\begin{aligned}&\vec{x} = (\rho\cos\varphi, \rho\sin\varphi, z), \\ &\vec{e}_\rho = (\cos\varphi, \sin\varphi, 0),\ \ \vec{e}_\varphi = (-\sin\varphi, \cos\varphi, 0),\ \ \vec{e}_z = (0,0,1).\end{aligned} \tag{C.2.1}$$

In cylindrical coordinates the gradient of a scalar field $\Phi(\rho,\varphi,z)$ takes the form

$$\vec{\nabla}\Phi = \partial_\rho\Phi\ \vec{e}_\rho + \frac{1}{\rho}\partial_\varphi\Phi\ \vec{e}_\varphi + \partial_z\Phi\ \vec{e}_z, \tag{C.2.2}$$

and the Laplacian is given by

$$\Delta\Phi = \frac{1}{\rho}\partial_\rho(\rho\partial_\rho\Phi) + \frac{1}{\rho^2}\partial_\varphi^2\Phi + \partial_z^2\Phi. \tag{C.2.3}$$

A general vector field can be written as

$$\vec{E}(\vec{x}) = E_\rho(\rho,\varphi,z)\vec{e}_\rho + E_\varphi(\rho,\varphi,z)\vec{e}_\varphi + E_z(\rho,\varphi,z)\vec{e}_z. \tag{C.2.4}$$

The general expression for the divergence of a vector field in cylindrical coordinates takes the form

$$\vec{\nabla}\cdot\vec{E} = \frac{1}{\rho}\partial_\rho(\rho E_\rho) + \frac{1}{\rho}\partial_\varphi E_\varphi + \partial_z E_z, \tag{C.2.5}$$

and the curl is given by

$$\vec{\nabla}\times\vec{E} = \left[\frac{1}{\rho}\partial_\varphi E_z - \partial_z E_\varphi\right]\vec{e}_\rho + \left[\partial_z E_\rho - \partial_\rho E_z\right]\vec{e}_\varphi + \left[\frac{1}{\rho}\partial_\rho(\rho E_\varphi) - \frac{1}{\rho}\partial_\varphi E_\rho\right]\vec{e}_z. \tag{C.2.6}$$