Aspects of Classical Mechanics

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Chapter 1

Space, Time, and Science

Physics is an empirical science, i.e. our knowledge of natural phenomena results from observation. In particular, one performs controlled experiments to measure physical quantities as accurately as possible. The value of a physical quantity is given in specified units, e.g. the basic unit of time is the second, and distances are measured in meters. However, physics is much more than a collection of experimental results. We use the results of experiments to construct theories that describe the data mathematically, for example, by establishing mathematical equations between various physical quantities. Again, a theory is much more than a collection of mathematical equations. It is a consistent self-contained mathematical framework describing a large complex of natural phenomena. In particular, the theory allows us to predict the results of further experiments, by working out the mathematical consequences of the basic equations. By actually performing the new experiments we can test the theory. In fact, a framework which is mathematically consistent may still be wrong because it does not describe the true behavior of Nature correctly. Before we call a mathematical framework a physical theory, it must pass numerous experimental tests. In principle, this testing never comes to an end, because we can never verify the theory completely. There may always be a new experiment that the theory cannot describe. Then the theory would be falsified and should be modified or replaced completely. This in principle never ending interaction between experiment and theory is called the scientific method. The big mystery — and perhaps one of the most exciting aspects of physics — is that this method actually works extraordinarily well. The Universe — although it offers an enormous spectrum of phenomena ranging from the physics of elementary particles to the behavior of the entire cosmos — seems to be governed by very few elementary laws of Nature. The goal of physical science is to discover those laws and work out their consequences. Physics is a very challenging enterprise because we certainly don't know all the laws of Nature yet. Perhaps we never will. However, thanks to previous generations of scientists, we know a lot already, and we can build our new theories on the ones they have developed.

The oldest physical theory in the sense described above is classical mechanics. It was initiated by Galileo Galilei, who was first to perform experiments in the modern sense, and formulated by Newton in the seventeenth century. Since then, i.e. for about 300 years, the theory has been tested experimentally, and it has passed almost all these tests incredibly well. Classical mechanics correctly describes the motion of celestial bodies, the mechanical phenomena in our everyday life, as well as macroscopic dynamics of rigid bodies, gases, and fluids. In particular, it is of immense importance for technical applications. In addition, classical mechanics is the prototype of any physical theory, and it already contains many of the basic concepts of the more modern theories that go beyond it. Classical mechanics describes the motion of macroscopic bodies, i.e. their position in space as a function of time. Once we specify the initial positions and velocities of the bodies, classical mechanics allows us to predict their motion in the future. In classical mechanics we make a few basic assumptions about space and time — the most fundamental categories of the physical world. Space is 3-dimensional and Euclidean, i.e. Pythagoras' theorem holds. This was actually tested experimentally by Gauss who measured large triangles with the summits of mountains as their corners. Time is 1-dimensional and absolute, i.e. it flows independent of the motion of the observer. Space and time are so basic that we can presently not define them from more elementary concepts. Instead, we define them by an operational procedure. For example, time is what we read off from a clock, except that ordinary wrist-watches may not satisfy the high standards of today's high precision atomic clocks.

Experiments at the end of the nineteenth and the beginning of the twentieth century have shown that the assumptions we make about space and time in classical physics are not completely justified. This led Einstein to develop special relativity in which 3-dimensional space and 1-dimensional time are merged to 4-dimensional space-time. In general relativity even space itself is no longer Euclidean and may, in fact, be curved. Relativistic effects are not noticeable in our everyday life. Only if we could move around with almost the velocity of light, we would experience the relativistic properties of space-time. In particular, at the moderate velocities we encounter in our life, relativistic effects are completely negligible, and the predictions of relativistic theories practically agree with the ones of ordinary non-relativistic Newtonian classical mechanics. The existence of relativistic effects does not mean that classical mechanics is wrong. However, it is incomplete and some of its assumptions about space and time are too restrictive under extreme conditions.

Classical mechanics is also incomplete in another respect. In classical mechanics we assume that we can measure a particle's position and velocity with, at least in principle, arbitrarily high precision. In fact, we need to specify both position and velocity at an initial time before we can predict the motion of a particle in the future. As various experiments performed at the beginning of the twentieth century have demonstrated, it is indeed impossible to simultaneously measure position and velocity of a particle with arbitrarily high precision. This is intimately related to the quantum nature of physical phenomena, and has led Heisenberg and others to the formulation of an extended version of mechanics — quantum mechanics. Again, as far as our everyday experience with macroscopic bodies is concerned, the predictions of quantum mechanics are practically indistinguishable from those of classical mechanics. Only when we enter the microscopic world of atoms, atomic nuclei, or elementary particles, a classical description turns out to be incomplete, and quantum mechanics must be used instead.

When we study classical mechanics we deal with non-relativistic and non-quantum physics. This is not the whole story, but it still covers an enormous spectrum of physical phenomena, in particular, the ones most relevant for our everyday life and for technical applications. In addition, classical mechanics is a consistent, beautiful mathematical framework, and it should be a pleasure to become acquainted with it in every detail.

Classical mechanics (as well as relativity theory and quantum mechanics) offer a general framework to describe natural phenomena. However, before we can make predictions, we need to specify the forces acting between the various particles involved. At present we know four fundamental forces: the gravitational force, the electromagnetic force, as well as the weak and strong forces responsible for radioactive decay and the binding of atomic nuclei. The latter two act on microscopic scales and are of no direct concern in classical physics. The description of the electromagnetic force is the subject of electrodynamics. The gravitational force is the only fundamental force we deal with in classical mechanics. Still, the other fundamental forces have indirect effects. They lead to the binding of rigid bodies or the formation of fluids, or they may cause friction.

To summarize, classical mechanics is a theory describing the motion of macroscopic bodies based on the assumption that space is Euclidean and time is absolute. It deals with the fundamental force of gravity, while the other three fundamental forces enter only indirectly.

When we study classical mechanics, and, in particular, when we want to solve practical problems, we often consider idealized situations. For example, we approximate an extended body by a point particle or we assume that an extended body is perfectly rigid. Such simplifications are necessary in order to make the problem tractable, and it is important to isolate the relevant aspects of a problem from secondary effects that may be neglected.

1.1 The Cube of Physics

In order to orient ourselves in the space of physical theories, let us consider what one might call the "cube of physics". Classical mechanics — the subject of this course — has its well-deserved place in the space of all theories. Theory space can be spanned by three axes labelled with the three most fundamental constants of Nature: Newton's gravitational constant

$$G = 6.6720 \times 10^{-11} \mathrm{kg}^{-1} \mathrm{m}^3 \mathrm{sec}^{-2}, \quad (1.1.1)$$

the velocity of light

$$c = 2.99792456 \times 10^8 \mathrm{m\ sec}^{-1}, \quad (1.1.2)$$

(which would deserve the name Einstein's constant), and Planck's quantum

$$h = 6.6205 \times 10^{-34} \mathrm{kg\ m}^2 \mathrm{sec}^{-1}. \quad (1.1.3)$$

Actually, it is most convenient to label the three axes by G, $1/c$, and h. For a long time it was not known that light travels at a finite speed or that there is quantum mechanics. In particular, Newton's classical mechanics corresponds to $c = \infty \Rightarrow 1/c = 0$ and $h = 0$, i.e. it is non-relativistic and non-quantum, and it thus takes place along the G-axis. If we also ignore gravity and put $G = 0$ we are at the origin of theory space doing Newton's classical mechanics but only with non-gravitational forces. Of course, Newton realized that in Nature $G \neq 0$, but he couldn't take into account $1/c \neq 0$ or $h \neq 0$. Maxwell and the other fathers of electrodynamics had c built into their theory as a fundamental constant, and Einstein realized that Newton's classical mechanics needs to be modified to his theory of special relativity in order to take into account that $1/c \neq 0$. This opened a new dimension in theory space which extends Newton's line of classical mechanics to the plane of relativistic theories. When we take the limit $c \to \infty \Rightarrow 1/c \to 0$ special relativity reduces to Newton's non-relativistic classical mechanics. The fact that special relativity replaced non-relativistic classical mechanics does not mean that Newton was wrong. Indeed, his theory emerges from Einstein's in the limit $c \to \infty$, i.e. if light would travel with infinite speed. As far as our everyday experience is concerned this is practically the case, and hence for these purposes Newton's theory is sufficient. There is no need to convince a mechanical engineer to use Einstein's theory because her airplanes are not designed to reach speeds anywhere close to c. Once special relativity was discovered, it became obvious that there must be a theory that takes into account $1/c \neq 0$ and $G \neq 0$ at the same time. After years of hard work, Einstein was able to construct this theory — general relativity — which is a relativistic theory of gravity. The G-$1/c$-plane contains classical (i.e. non-quantum) relativistic and non-relativistic physics. Classical electrodynamics fits together naturally with general relativity, but can also be considered in the absence of gravity, by assuming $G = 0$ but $1/c \neq 0$.

A third dimension in theory space was discovered by Planck who started quantum mechanics and introduced the fundamental action quantum h. When we put $h = 0$ quantum physics reduces to classical physics. Again, the existence of quantum mechanics does not mean that classical mechanics is wrong. It is, however, incomplete and should not be applied to the microscopic quantum world. In fact, classical mechanics is entirely contained within quantum mechanics as the limit $h \to 0$, just like it is the $c \to \infty$ limit of special relativity. Quantum mechanics was first constructed non-relativistically (i.e. by putting $1/c = 0$). When we allow $h \neq 0$ as well as $1/c \neq 0$ (but put $G = 0$) we are doing relativistic quantum physics. This is where the quantum version of classical electrodynamics — quantum electrodynamics (QED) — is located in theory space. Also the entire standard model of elementary particle physics which includes QED as well as its analog for the strong force — quantum chromodynamics (QCD) — is located there. Today we know that there must be a consistent physical theory that allows $G \neq 0$, $1/c \neq 0$, and $h \neq 0$ all at the same time. However, this theory of relativistic quantum gravity has not yet been found, although there are some promising attempts using, for example, string theory.

1.2 Euclidean Space and the Notion of a Vector

In classical mechanics we work in a 3-dimensional Euclidean space, and indeed our Universe is to very high accuracy Euclidean. For simplicity, let us begin with a 2-dimensional Euclidean space — a simple plane. Since the plane is 2-dimensional, we need two real numbers to define the position of a point. These two numbers are the coordinates of a point in the plane. For example, we may choose Cartesian coordinates, i.e. we pick an origin and two orthogonal coordinate axes. Each point in the plane is then associated with a pair of coordinates (x, y).

Since the plane is Euclidean, we can use Pythagoras' theorem to calculate the distance r between the point and the origin

$$r = \sqrt{x^2 + y^2}. \tag{1.2.1}$$

In order to find our point, we may start out at the origin and go the distance r in the appropriate direction. A quantity that has both direction and magnitude is called a vector. The displacement of the point (x, y) from the origin is given by the vector

$$\vec{r} = (x, y) \tag{1.2.2}$$

Here x and y are the two components of the vector $\vec{r}$ and r is its magnitude. We are not forced to use Cartesian coordinates. In fact, we may pick any coordinates that seem appropriate. Sometimes it is advantageous to use polar coordinates r and φ, where φ is an angle that specifies the direction of our vector. Cartesian and polar coordinates are related by

$$x = r\cos\varphi, \;\; y = r\sin\varphi. \tag{1.2.3}$$

Hence, we may write the vector as

$$\vec{r} = (x, y) = (r\cos\varphi, r\sin\varphi) = r(\cos\varphi, \sin\varphi). \tag{1.2.4}$$

Here we have used a simple rule of vector algebra. We have multiplied a vector by a real number (r in this case) by multiplying each of its components by this number. In general we may write

$$\lambda(x, y) = (\lambda x, \lambda y), \lambda \in \mathbb{R}. \tag{1.2.5}$$

The product of a vector with a real number is another vector with the same (or opposite) direction but with in general different magnitude. In our example, we have written the vector $\vec{r}$ as a product of its magnitude r and another vector

$$\vec{e}_r = (\cos\varphi, \sin\varphi). \tag{1.2.6}$$

What is the magnitude of $\vec{e}_r$? We write

$$|\vec{e}_r| = \sqrt{\cos^2\varphi + \sin^2\varphi} = \sqrt{1} = 1, \tag{1.2.7}$$

i.e. the vector $\vec{e}_r$ has unit length — it is a so-called unit-vector. The unit-vector $\vec{e}_r$ specifies the direction of the vector r, and $r = |\vec{r}|$ specifies its magnitude. It is convenient to introduce the unit vectors parallel to the Cartesian coordinate axes

$$\begin{aligned} \vec{e}_x &= (\cos 0, \sin 0) = (1, 0), \\ \vec{e}_y &= (\cos\frac{\pi}{2}, \sin\frac{\pi}{2}) = (0, 1). \end{aligned} \tag{1.2.8}$$

Now we may represent the vector $\vec{r}$ as

$$\vec{r} = (x, y) = x(1, 0) + y(0, 1) = x\vec{e}_x + y\vec{e}_y. \tag{1.2.9}$$

Here we have used another important rule of vector algebra. The sum of two vectors is another vector which is given by component-wise addition

$$\vec{r}_1 + \vec{r}_2 = (x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2). \tag{1.2.10}$$

What is the displacement between two arbitrary points in the plane? The displacement is given by the difference vector

$$\vec{r} = \vec{r}_1 - \vec{r}_2 = (x_1, y_1) - (x_2, y_2) = (x_1 - x_2, y_1 - y_2). \tag{1.2.11}$$

The distance between the two points described by $\vec{r}_1$ and $\vec{r}_2$ is given by

$$r = |\vec{r}| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1.2.12}$$

The distance is obviously not a vector. In fact, it is known as a scalar. Note that the components of a vector are not scalars, since, by definition, a scalar remains unchanged under spatial rotations, while the components of a vector are changing.

Not all geometries are Euclidean. For example, we can work on the surface of a sphere, which is also 2-dimensional but not flat. Obviously, Pythagoras' theorem does not apply to a rectangular triangle we draw on the surface of the sphere, as long as we define the distance as the length of the shortest geodesics on the sphere. Still, we may embed the 2-dimensional surface of the sphere in a 3-dimensional space, just as the earth is embedded in the space of classical mechanics. The notion of a vector readily extends to three (and, in fact, to any higher) dimensions. the position of a point in 3-dimensional space is characterized by the vector

$$\vec{r} = (x, y, z) \tag{1.2.13}$$

of magnitude

$$r = |\vec{r}| = \sqrt{x^2 + y^2 + z^2}. \tag{1.2.14}$$

We may also choose spherical coordinates r, θ, and φ. They are related to Cartesian coordinates by

$$x = r \sin\theta \cos\varphi, \; y = r \sin\theta \sin\varphi, \; z = r \cos\theta. \tag{1.2.15}$$

Again, we may construct a unit-vector that specifies the direction of $\vec{r}$

$$\vec{r} = r\vec{e}_r, \; \vec{e}_r = (\sin\theta \cos\varphi, \sin\theta \sin\varphi, \cos\theta), \tag{1.2.16}$$

which indeed has unit length because

$$|\vec{e}_r| = \sqrt{\sin^2\theta\cos^2\varphi + \sin^2\theta\sin^2\varphi + \cos^2\theta} = \sqrt{\sin^2\theta + \cos^2\theta} = 1. \quad (1.2.17)$$

Let us also define

$$\vec{e}_x = (1,0,0),\ \vec{e}_y = (0,1,0),\ \vec{e}_z = (0,0,1), \quad (1.2.18)$$

such that

$$\vec{r} = (x,y,z) = x\vec{e}_x + y\vec{e}_y + z\vec{e}_z. \quad (1.2.19)$$

Distances (as any other lengths) are measured in meters. It is necessary to express a dimensionful quantity (like a length) in an appropriate unit. The unit is in principle arbitrary. However, it is important to make a generally accepted reproducible choice. It is impossible to measure physical quantities to arbitrary precision, and it makes no sense to quote a result to more than its significant figures. For examples, if one measures a length with an ordinary meter stick, the accuracy is in the millimeter range.

1.3 Time, Velocity, and Acceleration

The basic unit of time is the second. Time is such a fundamental quantity that we can only define it by an operational procedure: time is what we read off from a clock. Time is 1-dimensional and we experience its constant flow from the past to the future. At present, physics does not offer a completely satisfactory explanation for why this is the case. Let us consider the motion of a body, say a baseball through the air. We idealize the ball as a point particle (i.e. we neglect the fact that it may rotate) and we describe its position in space by a vector $\vec{r}$. Obviously, the position changes with time, i.e. the vector is a function of time

$$\vec{r}(t) = (x(t), y(t), z(t)), \quad (1.3.1)$$

which is specified by the time-dependence of each of its components. The velocity of the particle is the rate of change of position, i.e. it is the derivative of the position vector

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \left(\frac{dx(t)}{dt}, \frac{dy(t)}{dt}, \frac{dz(t)}{dt}\right). \quad (1.3.2)$$

Velocity is again a vector, it has direction and magnitude. The magnitude of the velocity is a scalar — the so-called speed

$$v(t) = |\vec{v}(t)| = \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2}. \quad (1.3.3)$$

To illustrate velocity, one may consider a space-time diagram for 1-dimensional motion. The x-component of the velocity vector is given by

$$v_x(t) = \frac{dx(t)}{dt} = \lim_{\Delta t \to 0} \frac{x(t + \Delta t) - x(t)}{\Delta t}. \tag{1.3.4}$$

It corresponds to the slope of the curve in the space-time diagram. Velocity is a quantity derived from position and time. We need not invent a new independent unit for it. Speed is measured in meters per second.

Acceleration is the rate of change of velocity

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2}, \tag{1.3.5}$$

and hence also a vector. In classical mechanics, position, velocity, and acceleration play a central role, while higher order derivatives of the position vector do not appear in the theoretical framework.

Now let us consider motion with constant (i.e. time-independent) acceleration. This applies, for example, to falling bodies close to the surface of the earth. We obtain the velocity by integrating the acceleration

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} dt' \ \vec{a} = \vec{v}(t_0) + (t - t_0)\vec{a}, \tag{1.3.6}$$

and the position by integrating the velocity

$$\begin{aligned} \vec{r}(t) &= \vec{r}(t_0) + \int_{t_0}^{t} dt' \ \vec{v}(t') = \vec{r}(t_0) + \int_{t_0}^{t} dt' \ [\vec{v}(t_0) + (t - t_0)\vec{a}] \\ &= \vec{r}(t_0) + (t - t_0)\vec{v}(t_0) + \frac{1}{2}(t - t_0)^2\vec{a}. \end{aligned} \tag{1.3.7}$$

Let us choose $t_0 = 0$ and let us denote $\vec{v}(t_0) = \vec{v}(0) = \vec{v}_0$ and $\vec{r}(t_0) = \vec{r}(0) = \vec{r}_0$ such that

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t, \ \ \vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2. \tag{1.3.8}$$

Specializing to 1-dimensional motion we obtain

$$v_x(t) = v_0 + at, \ \ x(t) = x_0 + v_0 t + \frac{1}{2}at^2. \tag{1.3.9}$$

Velocity is always measured relative to a given coordinate system. Observers moving at different velocities measure different relative velocities of the same moving body. If two bodies move with the velocities $\vec{v}_1$ and $\vec{v}_2$, their relative velocity is

$$\vec{v} = \vec{v}_1 - \vec{v}_2. \tag{1.3.10}$$

For example, if a train moves at velocity v_1 and somebody in the train moves with velocity v relative to the train, then he moves with velocity $v_2 = v_1 + v$ relative to the observer at rest. This kind of relativity already occurs in Newton's classical mechanics, and is different from Einstein's relativity theory. In fact, if somebody in the train switches on a lamp emitting light relative to the train at velocity c, the observer at rest does not observe that light at velocity $v_1 + c$ (as non-relativistic classical mechanics would predict). In fact, he observes the same velocity c. In classical mechanics, we will always compute relative velocities simply by building vector sums as above. This is correct as long as the velocities involved are small compared to the velocity of light c.

1.4 Falling Bodies close to the Earth's Surface

Any massive body generates its own gravitational field. Very massive bodies like the earth generate a particularly strong gravitational field. Let us consider the earth as spherical with radius $R = 6400$ km and mass $M = 6 \times 10^{24}$ kg. Then, according to Newton's laws, the acceleration of the body falling in the gravitational field of the earth is given by

$$\vec{a}(t) = -\frac{GM}{r(t)^2}\vec{e}_r(t). \tag{1.4.1}$$

Here $\vec{r}$ is the position vector of the body (with the earth's center at the origin of the coordinate system), and $G = 6.67 \times 10^{-11}\text{m}^3/\text{kg s}^2$ is Newton's gravitational constant. For the moment, let us take the above formula as given. We will discuss it in much more detail later.

Now we are interested in the acceleration close to the earth's surface. In a neighborhood of an arbitrarily chosen z-axis, the earth is almost flat. Using spherical coordinates, we may then write

$$\vec{r} = r\sin\theta\cos\varphi\ \vec{e}_x + r\sin\theta\sin\varphi\ \vec{e}_y + r\cos\theta\ \vec{e}_z. \tag{1.4.2}$$

For points close to the z-axis we have $\theta \approx 0$ such that $\sin\theta \approx 0$ and $\cos\theta \approx 1$. Furthermore, for points a small height h above the surface of the earth, we have $r = R + h \approx R$ (since $h \ll R$) such that $\vec{r} \approx R\vec{e}_z$ and hence

$$\vec{a}(t) = -\frac{GM}{R^2}\vec{e}_z = -g\vec{e}_z. \tag{1.4.3}$$

The magnitude of the gravitational acceleration near the surface of the earth is hence given by

$$g = \frac{GM}{R^2} = \frac{6.67 \times 10^{-11}\text{m}^3/(\text{kg s}^2)\ 6 \times 10^{24}\text{kg}}{(6.4 \times 10^6)^2\text{m}^2} = 9.8\text{m/s}^2. \tag{1.4.4}$$

Chapter 2

Motion in Two Dimensions

In this chapter, we will discuss special cases of motion in two dimensions, in particular, the motion of a projectile as well as circular motion.

2.1 Motion of Projectiles

We are interested in the motion of a projectile (i.e. a thrown object) subject to gravity close to the earth's surface. To a very good approximation this is a problem with constant acceleration. Hence, we may describe the position of the projectile by

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0\ t + \frac{1}{2}\vec{a}\ t^2. \tag{2.1.1}$$

Let us choose a vertical z-axis such that $\vec{a} = -g\vec{e}_z$, and let the horizontal motion be restricted to the x-direction. Then we have a 2-dimensional problem which is described by

$$x(t) = x_0 + v_{x0}t,\ \ z(t) = z_0 + v_{z0}t - \frac{1}{2}gt^2. \tag{2.1.2}$$

Let us assume that the projectile starts at the origin, i.e. $x_0 = z_0 = 0$. We may then eliminate time from the equations

$$t = \frac{x(t)}{v_{x0}}, \tag{2.1.3}$$

such that

$$z(t) = \frac{v_{z0}}{v_{x0}}x(t) - \frac{g}{2v_{x0}^2}x(t)^2. \tag{2.1.4}$$

We may now view z as a function of x and obtain

$$z(x) = \frac{v_{z0}}{v_{x0}} x - \frac{g}{2v_{x0}^2} x^2. \tag{2.1.5}$$

This is the equation of a parabola. If we throw the projectile, how far does it go in the x-direction? It reaches ground level at $z(x) = 0$ with

$$\frac{v_{z0}}{v_{x0}} x - \frac{g}{2v_{x0}^2} x^2 = 0 \ \Rightarrow \ x = 0 \text{ or } x = \frac{2v_{x0}v_{z0}}{g}. \tag{2.1.6}$$

Let us suppose that we can throw the projectile with a maximal initial speed v_0. How far in the x-direction can we then throw it, and under what angle α should we eject it in order to reach the largest distance? We have

$$v_{x0} = v_0 \cos\alpha, \ v_{z0} = v_0 \sin\alpha, \tag{2.1.7}$$

such that the distance in x is

$$x = \frac{2v_{x0}v_{z0}}{g} = \frac{2v_0^2 \sin\alpha \cos\alpha}{g} = \frac{v_0^2}{g} \sin(2\alpha). \tag{2.1.8}$$

This assumes its maximal value $x = v_0^2/g$ for $\sin(2\alpha) = 1$, i.e. $\alpha = \pi/4$, which corresponds to an angle of 45 degrees. How high is the projectile going? The maximal z-value is determined by

$$\frac{dz(x)}{dx} = \frac{v_{z0}}{v_{x0}} - \frac{g}{v_{x0}^2} x = 0 \ \Rightarrow \ x = \frac{v_{x0}v_{z0}}{g}, \tag{2.1.9}$$

and the corresponding height is

$$z(x) = \frac{v_{z0}}{v_{x0}} \frac{v_{x0}v_{z0}}{g} - \frac{1}{2} \frac{g}{v_{x0}^2} \frac{v_{x0}^2 v_{z0}^2}{g^2} = \frac{v_{z_0}^2}{2g}. \tag{2.1.10}$$

Finally, let us calculate the speed with which the projectile hits the ground

$$v_x(t) = \frac{dx(t)}{dt} = v_{x0}, \ v_z(t) = \frac{dz(t)}{dt} = v_{z0} - gt \ \Rightarrow$$
$$v(t) = \sqrt{v_x(t)^2 + v_z(t)^2} = \sqrt{v_{x0}^2 + (v_{z0}^2 - gt)^2}. \tag{2.1.11}$$

When the projectile hits the ground we have

$$z(t) = v_{z0} t - \frac{1}{2} g t^2 = 0 \ \Rightarrow \ gt = 2v_{z0}, \tag{2.1.12}$$

and the corresponding velocity

$$v(t) = \sqrt{v_{x0}^2 + (v_{z0}^2 - gt)^2} = \sqrt{v_{x0}^2 + v_{z0}^2} = v_0 \tag{2.1.13}$$

is hence equal to the initial velocity v_0.

2.2 Circular Motion

The motion along a circle even at constant speed is accelerated motion because the velocity vector is always changing its direction. Let us describe this motion in polar coordinates. Again, we have a 2-dimensional problem. Motion on a circle means $r(t) = r$. The polar angle φ changes linearly with time

$$\varphi(t) = \omega t, \tag{2.2.1}$$

where ω is the constant angular velocity. We then obtain

$$x(t) = r(t) \cos \varphi(t) = r \cos(\omega t), \ y(t) = r(t) \sin \varphi(t) = r \sin(\omega t). \tag{2.2.2}$$

Hence, the corresponding velocity takes the form

$$v_x(t) = \frac{dx(t)}{dt} = -r\omega \sin(\omega t), \ v_y(t) = \frac{dy(t)}{dt} = r\omega \cos(\omega t), \tag{2.2.3}$$

and the speed is given by

$$v(t) = \sqrt{v_{x0}^2 + v_{z0}^2} = r\omega, \tag{2.2.4}$$

which is indeed constant. Let us also calculate the acceleration

$$a_x(t) = \frac{dv_x(t)}{dt} = -r\omega^2 \cos(\omega t), \ a_y(t) = \frac{dv_y(t)}{dt} = -r\omega^2 \sin(\omega t), \tag{2.2.5}$$

which can be summarized as

$$\vec{a}(t) = -\omega^2 \vec{r}(t). \tag{2.2.6}$$

This implies that the acceleration vector points to the center of the circle. The magnitude of the acceleration is given by

$$a(t) = \sqrt{a_x(t)^2 + a_y(t)^2} = \omega^2 r = \frac{v^2}{r}, \tag{2.2.7}$$

which is again independent of time.

Chapter 3

Mass, Force, and Newton's Laws

So far we have described the motion of various bodies by their position vector as a function of time, as well as by their velocity and acceleration. That subject is called kinematics. Now we will ask what causes motion, and, in particular, what causes change of velocity — namely acceleration. We will find that acceleration of bodies is caused by external influences, known as forces. When we describe motion using the concept of force, we enter the subject of dynamics. We will find that the same external influence (i.e. the same force) affects different bodies in different ways. Depending on its inertia, a body is accelerated more or less strongly by a given external force. The measure of inertia is mass (or inertial mass). It is measured in a new basic unit — the kilogram (kg). In this chapter, we will learn about the relation between force, mass, and acceleration (Newton's second law), about various (more or less fundamental) force laws, and about equations of motion, which are mathematically represented by differential equations.

3.1 The Concepts of Mass and Force

Let us begin with a discussion of Newton's first law: a body left undisturbed maintains a constant velocity. Apparently, this law seems to be violated in situations of our everyday life, because, after being put in motion, things do come to rest after a while. This is because it is difficult to completely isolate a body from external influences. In practice, usually there are friction forces that cause a moving body to slow down. Under ideal conditions (e.g. in a vacuum, or in outer space) a body put in motion maintains its velocity. To cause a change of velocity — in other words, in order to accelerate the body — one needs to influence it from outside.

Let us consider a spring that has a length x_0 when it is relaxed. Now we compress the spring and we place a body in front of it. Once we release the spring, it pushes the body forward, which then starts moving. In other words, the body is accelerated by the spring with some amount a_1. Next, we repeat the experiment with some other body, but we compress the spring by exactly the same amount. The second body is, in general, accelerated by a different amount a_2, because it has a different amount of inertia. When we repeat our experiment again with the same bodies, but now compressing the spring further, both will be accelerated more strongly. However, the ratio of the two accelerations a_1/a_2 will remain the same as before. In other words, this ratio is independent of the external influence (caused by the spring), it is an intrinsic property of the two bodies. This constant ratio determines the mass ratio of the two bodies

$$\frac{a_1}{a_2} = \frac{m_2}{m_1}. \tag{3.1.1}$$

We compare the masses m_1 and m_2 of the two bodies (their amount of inertia) by comparing the accelerations a_1 and a_2 caused by the same external influence. A small mass is less inert and hence is accelerated more strongly by a given external influence. In order to set a standard of mass, a piece of matter has been deposited near Paris a long time ago. The mass of this piece of matter has been defined (!) to be $m_1 = 1$ kg. If we want to measure the mass m_2 of some other body, we should hence go to Paris, put both masses under the same external influence, and compare their accelerations, such that

$$m_2 = \frac{a_1}{a_2} m_1 = \frac{a_1}{a_2} \text{kg}. \tag{3.1.2}$$

Fortunately, in order to save the trip to Paris, identical copies of the standard kilogram exist everywhere around the world. We can turn eq.(3.1.1) around and write

$$m_1 a_1 = m_2 a_2. \tag{3.1.3}$$

This product is the same for different bodies which are affected by the same external influence, and hence characteristic of the external influence itself. The product ma is a very important quantity in Newtonian mechanics, known as the force $F = ma$. The unit of force is one Newton, $\text{N} = \text{kg m/s}^2$.

So far we have restricted ourselves to 1-dimensional motion. However, we know that, in general, acceleration is a vector. Mass, on the other hand, is a scalar (it does not change under spatial rotations), and hence force must also be a vector

$$\vec{F} = m\vec{a}. \tag{3.1.4}$$

This is Newton's second law. Force has both magnitude and direction, e.g. a spring can push or pull an object in different directions. Just like velocities or accelerations, forces add up vectorially.

When Nobel laureate Frank Wilczek teaches classical mechanics, he also stresses what he calls "Newton's zeroth law": every mechanical body possesses a non-zero mass. Indeed, classical mechanics can only be applied to massive bodies. As we know today, Newton's zeroth law does not apply to all physical objects. For example, photons — the quanta of light — are massless and cannot be accelerated beyond the velocity c. Mass is a fundamental concept. Indeed, within the framework of Newtonian mechanics one cannot answer questions concerning the origin of mass. The mass of the matter that surrounds us is almost entirely concentrated in atomic nuclei which consist of protons and neutrons. These particles, in turn, consist of quarks and gluons which are permanently confined together by the strong interactions. Indeed, the origin of mass is the strong interaction energy of elementary quarks and gluons.

3.2 Various Force Laws and Newton's Third Law

We distinguish four fundamental forces in Nature: gravity, electromagnetism, as well as the weak and strong nuclear forces. These four forces are described by fundamental force laws. All other forces are more or less direct consequences of the fundamental forces. A spring, for example, gets its tension from the interatomic forces in its interior, which are of electromagnetic origin. Still, we can describe the force of a spring by an effective force law — Hooke's law — which is not very fundamental (because a spring is not a very fundamental object), but it describes the force of the spring much more efficiently than the superposition of numerous fundamental electromagnetic forces between individual atoms.

The fundamental law describing the electric force that a charged particle with charge q_1 experiences in the presence of another charged particle of charge q_2 is Coulomb's law

$$\vec{F} = \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2). \tag{3.2.1}$$

Here $\vec{r}_1$ and $\vec{r}_2$ are the position vectors of the two particles. When both charges are positive (or both negative), the force pushes particle 1 away from particle 2, the force is repulsive. If, on the other hand, the charges have opposite signs, the force is attractive.

Now let us consider the force $\vec{F}'$ that particle 2 experiences due to the presence of particle 1. Of course, this is also given by Coulomb's law, except that we now must exchange 1 and 2, such that

$$\vec{F}' = \frac{q_2 q_1}{|\vec{r}_2 - \vec{r}_1|^3}(\vec{r}_2 - \vec{r}_1) = -\vec{F}. \tag{3.2.2}$$

This is an example of Newton's third law. The force $\vec{F}'$ that particle 2 experiences due to the presence of particle 1 is of the same magnitude, but of opposite direction, as the force $\vec{F}$ that particle 1 feels due to the presence of particle 2. This is true not only for electric forces, but for all forces we encounter in classical mechanics. Newton's third law is sometimes summarized as "actio equals reactio". It does not make sense to add up $\vec{F}$ and $\vec{F}'$ to a total force that vanishes, because the two forces act on different particles. The force $\vec{F}$ influences particle 1, while the force $\vec{F}'$ influences particle 2.

Next, we consider the fundamental force of gravity. The force law is very similar to Coulomb's law and is given by

$$\vec{F} = -\frac{G m_1 m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2). \tag{3.2.3}$$

Now $\vec{F}$ is the gravitational force that particle 1 with mass m_1 experiences due to the presence of particle 2 of mass m_2. Unlike the charge, the mass of a particle is always positive. Hence, due to the overall minus-sign, gravity is always attractive. It is very remarkable that the mass determines the strength of the gravitational force, although it was initially introduced as a measure of inertia. A priori, one could have expected that the inertial mass differs from the gravitational mass that determines the strength of gravity. This is, however, not the case. The fact that the inertial and gravitational masses are identical is known as the equivalence principle, which is at the basis of general relativity. As a consequence of the equivalence principle, the acceleration experienced by particle 1 is independent of its mass

$$\vec{a}_1 = \frac{\vec{F}}{m_1} = -\frac{G m_2}{|\vec{r}_1 - \vec{r}_2|^3}(\vec{r}_1 - \vec{r}_2). \tag{3.2.4}$$

This is typical for gravity, but is not the case for any other force in Nature. Close to the earth's surface the force experienced by a massive body is given by

$$\vec{F} = -m g \vec{e}_z. \tag{3.2.5}$$

This force is called the weight. Hence, it is measured in N, not in kg. In section I.3 we computed $g = GM/R^2$, where M is the mass of the earth and R is earth's radius. In fact, there is a small additional contribution due to earth's rotation. The weight of an object is not an intrinsic property of the object, e.g. because it depends on the earth's mass and radius. If the same object is placed on the moon its weight gets smaller because the gravitational acceleration of the moon is smaller than g on earth. Still, the mass of an object is the same everywhere. Mass is an intrinsic property of the body, and it is of much more fundamental nature than the body's weight.

Finally, let us discuss a non-fundamental force — the one caused by a compressed (or expanded) spring. The tension of a spring is due to inter-atomic forces which are of electromagnetic nature. Still, on a macroscopic level we can describe the force by a simple formula — Hooke's law — that does not refer to the Coulomb forces present on microscopic scales. First, we consider a relaxed spring of equilibrium length x_0. When we attach a body to a relaxed spring, no force is exerted on it. Next, we compress the spring. Then there is a force pushing the body towards positive x. When we expand the spring beyond x_0, it pulls the object back towards smaller x. This behavior is described by Hooke's law

$$F = -k(x - x_0). \tag{3.2.6}$$

The positive parameter k is known as the spring constant. It is measured in $\text{N/m} = \text{kg/s}^2$. In practice, Hooke's law is valid only as long as $|x - x_0|$ is not too large. If we expand a spring too much, it will simply break. In this sense, Hooke's law is the force law for an idealized spring. Still, for not too large $|x - x_0|$ Hooke's law describes the force of the spring very accurately.

3.3 The Feebleness of Gravity

In our everyday life we experience gravity as a rather strong force because the enormous mass of the earth exerts an attractive gravitational force. The force of gravity acts universally on all massive objects. For example, two protons of mass M_p at a distance r exert a gravitational force

$$F_g = \frac{GM_p^2}{r^2} \tag{3.3.1}$$

on each other. However, protons carry an electric charge e and thus they also exert a repulsive electrostatic Coulomb force

$$F_e = \frac{e^2}{r^2} \tag{3.3.2}$$

on each other. Electromagnetic forces are less noticeable in our everyday life because protons are usually contained inside the nucleus of electrically neutral atoms, i.e. their charge e is screened by an equal and opposite charge $-e$ of an electron in the electron cloud of the atom. Still, as a force between elementary particles, electromagnetism is much stronger than gravity. The ratio of the electrostatic and gravitational forces between two protons is

$$\frac{F_e}{F_g} = \frac{e^2}{GM_p^2}. \tag{3.3.3}$$

The strength of electromagnetic interactions is determined by the fine-structure constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036}, \tag{3.3.4}$$

which is constructed from fundamental constants of Nature: the basic charge quantum e, the velocity of light c and Planck's quantum divided by 2π, $\hbar = h/2\pi$. The fine-structure constant is a dimensionless number and hence completely independent of arbitrary choices of physical units. Today we do not understand why it has the above experimentally determined value.

The strength of gravitational interactions is determined by Newton's constant

$$G = \frac{\hbar c}{M_{\text{Planck}}^2}, \tag{3.3.5}$$

and can be expressed in terms of the Planck mass M_{Planck} which is the highest energy scale relevant in elementary particle physics

$$M_{\text{Planck}} = 1.3014 \times 10^{19} M_p. \tag{3.3.6}$$

Remarkably, it is qualitatively understood why the proton mass M_p is a lot smaller than the Planck mass. This is a consequence of the property of asymptotic freedom of quantum chromodynamics (QCD) — the quantum field theory that describes the dynamics of quarks and gluons inside protons.

Using eq.(3.3.4), eq.(3.3.5), as well as eq.(3.3.6), eq.(3.3.3) can be cast in the form

$$\frac{F_e}{F_g} = \frac{e^2}{GM_p^2} = \alpha \frac{M_{\text{Planck}}^2}{M_p^2} = 1.236 \times 10^{36}. \tag{3.3.7}$$

Hence, as a fundamental force electromagnetism is very much stronger than gravity.

3.4 Equations of Motion as Differential Equations

Once we have specified a force law, we can predict the motion of a particle using Newton's second law. The motion is described by the position vector $\vec{r}(t)$ as a function of time. The force law specifies the force $\vec{F}(\vec{r})$ for any position $\vec{r}$ of the particle. Hence, we may write

$$\vec{F}(\vec{r}(t)) = m\vec{a}(t) = m\frac{d^2\vec{r}(t)}{dt^2}. \tag{3.4.1}$$

This is the equation of motion for the particle. Mathematically speaking, it is a differential equation: it relates the second derivative of the position vector to some function $(\vec{F}(\vec{r}))$ of the position vector itself. Let us illustrate this with a spring using Hooke's law, choosing the equilibrium position $x_0 = 0$, such that

$$m\frac{d^2x(t)}{dt^2} = F(x(t)) = -kx(t). \tag{3.4.2}$$

We rewrite this as

$$\frac{d^2x(t)}{dt^2} + \omega^2 x(t) = 0,\ \omega = \sqrt{\frac{k}{m}}. \tag{3.4.3}$$

This is one of the most important (and most simple) equations of motion in physics. Its general solution is given by

$$x(t) = A\cos(\omega t) + B\sin(\omega t), \tag{3.4.4}$$

where A and B must be determined from the initial conditions (i.e. the initial position and velocity of the particle). We have

$$\begin{aligned}\frac{dx(t)}{dt} &= -A\omega\sin(\omega t) + B\omega\cos(\omega t),\\ \frac{d^2x(t)}{dt^2} &= -A\omega^2\sin(\omega t) - B\omega^2\cos(\omega t) = -\omega^2 x(t),\end{aligned} \tag{3.4.5}$$

such that the equation of motion is indeed satisfied. It describes an oscillatory motion, and a system governed by this equation is known as a harmonic oscillator. The parameter ω is the angular frequency of the harmonic oscillator.

Let us also discuss another oscillator — a planar pendulum in the gravitational field near the earth's surface. The pendulum consists of a mass m suspended from the origin at a string of length l. The angle θ between the string and the z-axis measures the deviation from a vertical position. The speed of the pendulum is given by $ld\theta(t)/dt$, and its acceleration tangential to the circle along which it moves is given by

$$a(t) = l\frac{d^2\theta(t)}{dt^2}. \tag{3.4.6}$$

The component of the gravitational force in this direction is

$$F(\theta) = -mg\sin\theta, \tag{3.4.7}$$

such that Newton's second law takes the form

$$ml\frac{d^2\theta(t)}{dt^2} = ma(t) = F(\theta(t)) = -mg\sin\theta, \tag{3.4.8}$$

and hence

$$\frac{d^2\theta(t)}{dt^2} + \omega^2\sin\theta = 0, \ \omega = \sqrt{\frac{g}{l}}. \tag{3.4.9}$$

The equation is similar to the one of the harmonic oscillator, but its solution is much more complicated because $\sin\theta(t)$, and not just $\theta(t)$ itself, enters the equation. In general, a pendulum is an an-harmonic oscillator. For small angles θ, however, we may write $\sin\theta \approx \theta$ such that we then recover the equation of motion of the harmonic oscillator. In this approximation the period of the pendulum is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l}{g}}. \tag{3.4.10}$$

Next, let us consider a conical pendulum moving along a circle perpendicular to the direction of the gravitational force. Now the angle θ does not change with time, only the angle φ does. The speed of the bob is given by

$$v = l\sin\theta\frac{d\varphi(t)}{dt} = l\sin\theta\omega, \tag{3.4.11}$$

where ω is a constant angular velocity. The corresponding centripetal acceleration is

$$a = \frac{v^2}{l\sin\theta} = \omega^2 l\sin\theta. \tag{3.4.12}$$

The relevant component of the gravitational force that causes this motion is

$$F = mg\tan\theta, \tag{3.4.13}$$

such that Newton's second law now implies

$$m\omega^2 l\sin\theta = ma = F = mg\tan\theta \ \Rightarrow \ \omega = \sqrt{\frac{g}{l\cos\theta}}. \tag{3.4.14}$$

The corresponding period is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{l\cos\theta}{g}}. \tag{3.4.15}$$

Chapter 4

Energy

In this chapter a new concept is introduced, the concept of energy and its conservation. While position, velocity, acceleration, and force always refer to a given particle, in general energy is associated with a physical system as a whole. In particular, particles may exchange energy, while the total energy of a closed system of particles always remains the same. The conservation of energy is a very fundamental physical principle, related to the fact that the laws of Nature remain the same as time evolves. Energy may appear in many different forms. In classical mechanics we deal with kinetic and potential energy. Other forms of energy are, for example, heat, chemical energy, and radiation energy. The various forms of energy can be transformed into each other, but the total energy always remains the same. All fundamental forces are conservative, i.e. they conserve energy. Some non-fundamental forces seem to be non-conservative. For example, friction forces seem to lead to an energy decrease. This is, however, not true. Friction simply converts kinetic energy of a moving macroscopic body into heat, i.e. into chaotic, uncorrelated motion of the microscopic constituents of matter (i.e. atoms). This is what we want to avoid when we try to "save" energy. We try to prevent the transformation of highly organized useful forms of energy into chaotic less useful forms like heat.

4.1 Kinetic and Potential Energy

Let us consider 1-dimensional motion of some particle of mass m under a force $F(x)$ that may depend on the particle's position. The motion $x(t)$ of the particle is determined by Newton's second law. The corresponding equation of motion takes the form

$$F(x(t)) = ma(t) = m\frac{d^2x(t)}{dt^2}. \tag{4.1.1}$$

Now we introduce a quantity $V(x)$ defined by

$$F(x) = -\frac{dV(x)}{dx}. \tag{4.1.2}$$

Up to an arbitrary additive constant, we can determine $V(x)$ by integration of the force, i.e.

$$V(x) = V(x_0) - \int_{x_0}^{x} dx'\ F(x'). \tag{4.1.3}$$

The additive constant $V(x_0)$ is an arbitrary constant of integration. Given $V(x)$, we can rewrite the equation of motion as

$$ma(t) + \frac{dV(x)}{dx} = 0. \tag{4.1.4}$$

We multiply this equation by the velocity $v(t) = dx(t)/dt$, and we use $a(t) = dv(t)/dt$ such that

$$mv(t)\frac{dv(t)}{dt} + \frac{dV(x)}{dx}\frac{dx(t)}{dt} = 0. \tag{4.1.5}$$

This equation can be integrated. To see this we write

$$\frac{d}{dt}\left[\frac{1}{2}mv(t)^2 + V(x(t))\right] = mv(t)\frac{dv(t)}{dt} + \frac{dV(x)}{dx}\frac{dx(t)}{dt} = 0. \tag{4.1.6}$$

In other words, the expression in brackets is a constant independent of time. As time evolves, it remains unchanged. We say it is conserved. This constant is the energy

$$E = \frac{1}{2}mv(t)^2 + V(x(t)). \tag{4.1.7}$$

The first term $\frac{1}{2}mv(t)^2$ is the kinetic energy and the second term $V(x)$ is the potential energy. The kinetic energy depends on the velocity of the particle, while the potential energy depends on the particle's position. During the motion, the values of both the kinetic and the potential energy are varying. However, their sum (the total energy) remains the same (i.e. it is conserved).

Let us compute the potential energy for a particle attached to a spring. The force is then given by Hooke's law

$$F(x) = -k(x - x_0). \tag{4.1.8}$$

By integration we find

$$V(x) = V(x_0) + \frac{1}{2}k(x - x_0)^2. \tag{4.1.9}$$

It is natural to put the arbitrary constant $V(x_0)$ to zero. The potential energy is then described by a parabola. When the particle is at $x = x_0$ it has zero potential energy (i.e. the spring is relaxed and no energy is stored in it). Minima of the potential energy are stable equilibrium positions, because then the force vanishes, i.e.

$$F(x) = -\frac{dV(x)}{dx} = 0. \tag{4.1.10}$$

When the spring is compressed and the particle is put at $x = x_0 - A$, the potential energy

$$V(x) = V(x_0 - A) = \frac{1}{2}kA^2 \tag{4.1.11}$$

is stored in the spring. When the particle starts from $x = x_0 - A$ at rest, the spring will expand thus accelerating the particle. Thereby potential energy (stored in the spring) is converted into kinetic energy of the particle. The total energy is conserved and given by $\frac{1}{2}kA^2$. When the particle passes through $x = x_0$ its potential energy is zero (the spring is completely relaxed) and all its energy is now kinetic, i.e.

$$\frac{1}{2}mv^2 = \frac{1}{2}A^2 \;\Rightarrow\; v = \sqrt{\frac{k}{m}}A. \tag{4.1.12}$$

Later the spring expands and the kinetic energy is transformed back into potential energy until the particle turns around at $x = x_0 + A$.

Next, let us consider a particle in the gravitational field of the earth. We consider its motion along the z-direction. The gravitational force is given by

$$F(z) = -mg, \tag{4.1.13}$$

and the corresponding potential energy takes the form

$$V(z) = V(z_0) + mgz. \tag{4.1.14}$$

Again, we put $V(z_0) = 0$ such that the potential energy increases linearly with the height (say, above sea level). Now let us consider a landscape with the profile $z(x)$. This profile is directly proportional to the potential energy. Valleys are minima of potential energy and therefore positions of stable equilibrium, while summits are maxima of potential energy and hence positions of unstable equilibrium.

Next, let us consider motion in three dimensions. Then we must work with vectors, i.e.

$$\vec{F}(\vec{r}) = m\vec{a}(t). \tag{4.1.15}$$

In three dimensions the condition for the potential energy $V(\vec{r})$ is

$$\vec{F}(\vec{r}) = (F_x(\vec{r}), F_y(\vec{r}), F_z(\vec{r})) = -\left(\frac{dV(\vec{r})}{dx}, \frac{dV(\vec{r})}{dy}, \frac{dV(\vec{r})}{dz}\right). \tag{4.1.16}$$

Unlike in one dimension, in three dimensions it is not always possible to find a potential energy function $V(\vec{r})$ for a given force $\vec{F}(\vec{r})$. Still, for the most relevant forces (e.g. for gravity) it is possible. Such forces are called conservative.

Let us now take the scalar product of Newton's equation with the velocity vector, and let us assume that the force is conservative (i.e. there is a corresponding potential energy $V(\vec{r})$)

$$\begin{aligned} & m\vec{v}(t) \cdot \frac{d\vec{v}(t)}{dt} - \vec{F}(\vec{r}) \cdot \frac{d\vec{r}(t)}{dt} = 0 \ \Rightarrow \\ & m\left(v_x(t)\frac{dv_x(t)}{dt} + v_y(t)\frac{dv_y(t)}{dt} + v_z(t)\frac{dv_z(t)}{dt}\right) \\ & + \frac{dV(\vec{r})}{dx}\frac{dx(t)}{dt} + \frac{dV(\vec{r})}{dy}\frac{dy(t)}{dt} + \frac{dV(\vec{r})}{dz}\frac{dz(t)}{dt} = 0. \end{aligned} \tag{4.1.17}$$

Now we use the chain rule for differentiating a function of several variables

$$\frac{dV(\vec{r})}{dt} = \frac{dV(x,y,z)}{dt} = \frac{dV(\vec{r})}{dx}\frac{dx(t)}{dt} + \frac{dV(\vec{r})}{dy}\frac{dy(t)}{dt} + \frac{dV(\vec{r})}{dz}\frac{dz(t)}{dt}, \tag{4.1.18}$$

and we write

$$\frac{d}{dt}\left[\frac{1}{2}m\left(v_x(t)^2 + v_y(t)^2 + v_z(t)^2\right) + V(\vec{r})\right] = 0. \tag{4.1.19}$$

Again, we can read off the conserved energy

$$E = \frac{1}{2}m\left(v_x(t)^2 + v_y(t)^2 + v_z(t)^2\right) + V(\vec{r}) = \frac{1}{2}mv(t)^2 + V(\vec{r}). \tag{4.1.20}$$

The kinetic energy is proportional to the speed squared. Like speed, the energy (both kinetic and potential) is a scalar.

Finally, let us compute the potential energy of a particle of mass m at position $\vec{r}$ in the presence of another mass M at the origin. The gravitational force acting on the first particle is

$$\vec{F}(\vec{r}) = -\frac{GmM}{r^3}\vec{r} = \frac{GmM}{\sqrt{x^2+y^2+z^2}^3}(x,y,z). \tag{4.1.21}$$

We are trying to find a function $V(\vec{r}) = V(x,y,z)$ such that eq.(4.1.16) is satisfied. We can convince ourselves that

$$V(\vec{r}) = -\frac{GmM}{r} = -\frac{GmM}{\sqrt{x^2+y^2+z^2}} = -GmM(x^2+y^2+z^2)^{-1/2} \qquad (4.1.22)$$

obeys this condition. For example, we obtain

$$\frac{dV(\vec{r})}{dx} = -GmM\left(-\frac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}2x = \frac{GmM}{\sqrt{x^2+y^2+z^2}^3}x = -F_x(\vec{r}), \qquad (4.1.23)$$

as well as corresponding results for the other components.

We can use the expression for the potential energy to calculate the potential energy difference between a place at the earth's surface and some point in outer space, far away from the earth

$$\Delta V = V(R\vec{e}_r) - V(\infty) = -\frac{GmM}{R}. \qquad (4.1.24)$$

To reach such a point in outer space, one needs enough kinetic energy in order to compensate for the potential energy difference

$$\frac{1}{2}mv^2 + \Delta V = \frac{1}{2}mv^2 - \frac{GmM}{R} = 0 \;\Rightarrow$$
$$v = \sqrt{\frac{2GM}{R}} = \sqrt{2gR} \approx 40.000 \text{ km/h}. \qquad (4.1.25)$$

4.2 Work

As we have seen, the total energy of a physical system is conserved. Still, energy may be transferred from one part of the system to another one. For example, the potential energy stored in a compressed spring can be transformed into kinetic energy of a massive body attached to it. When this happens, we say that the force of the spring has done work on the particle, thereby changing its kinetic energy.

First, let us again consider the 1-dimensional case. Then work done by some force $F(x)$ acting on a particle that moves from an initial position x_i to a final position x_f is defined as

$$W = \int_{x_i}^{x_f} dx\; F(x). \qquad (4.2.1)$$

When the force is constant (independent of the position x) we get

$$W = F \int_{x_i}^{x_f} dx = F(x_f - x_i), \tag{4.2.2}$$

i.e., in that case, work is force times distance traveled. For general x-dependent forces we may write

$$W = \int_{x_i}^{x_f} dx\ F(x) = -\int_{x_i}^{x_f} dx\ \frac{dV(x)}{dx} = -V(x_f) + V(x_i), \tag{4.2.3}$$

i.e. work is the difference of potential energies. Next, we use energy conservation

$$\frac{1}{2}mv_i^2 + V(x_i) = E = \frac{1}{2}mv_f^2 + V(x_f) \tag{4.2.4}$$

to write

$$W = -V(x_f) + V(x_i) = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_i^2, \tag{4.2.5}$$

i.e. work changes the initial kinetic energy $\frac{1}{2}mv_i^2$ into the final kinetic energy $\frac{1}{2}mv_f^2$.

In the 3-dimensional case things are more complicated. Let us first consider a particle moving from an initial position $\vec{r}_i$ to a final position $\vec{r}_f$ under a constant ($\vec{r}$-independent) force $\vec{F}$. Then the work is given by the scalar product

$$W = \vec{F} \cdot (\vec{r}_f - \vec{r}_i). \tag{4.2.6}$$

Note that a force perpendicular to the displacement of the particle can not do work on the particle (i.e. it cannot change its kinetic energy) because the scalar product then vanishes.

Finally, let us consider the general case of an $\vec{r}$-dependent force. Then the work is defined as

$$W = \int_{\mathcal{C}} d\vec{r} \cdot \vec{F}(\vec{r}), \tag{4.2.7}$$

where the curve $\mathcal{C}$ is the particle's path connecting $\vec{r}_i$ with $\vec{r}_f$. We can use time to parametrize the integral as

$$W = \int_{t_i}^{t_f} dt\ \frac{d\vec{r}(t)}{dt} \cdot \vec{F}(\vec{r}). \tag{4.2.8}$$

If the force is conservative, i.e. if

$$\vec{F}(\vec{r}) = -\left(\frac{dV(\vec{r})}{dx}, \frac{dV(\vec{r})}{dy}, \frac{dV(\vec{r})}{dz}\right), \tag{4.2.9}$$

we can write

$$\begin{aligned} W &= -\int_{t_i}^{t_f} dt \left(\frac{dx(t)}{dt}\frac{dV(\vec{r})}{dx} + \frac{dy(t)}{dt}\frac{dV(\vec{r})}{dy} + \frac{dz(t)}{dt}\frac{dV(\vec{r})}{dz} \right) \\ &= -\int_{t_i}^{t_f} dt \ \frac{dV(\vec{r}(t))}{dt} = -V(\vec{r}(t_f)) + V(\vec{r}(t_i)) \\ &= -V(\vec{r}_f) + V(\vec{r}_i). \end{aligned} \tag{4.2.10}$$

The result is completely analogous to the 1-dimensional case, only the mathematics behind it is a bit more difficult.

Chapter 5

Systems of Particles

Many problems in physics involve more than just a single particle that is influenced by some external force. Instead, several particles may influence each other through various forces. These forces among the members of a system of particles are called internal forces. In addition, the system of particles may also be influenced from outside by so-called external forces. A closed system — i.e. one without external forces — can be characterized by a new conserved quantity — the total momentum. The total momentum of a system of particles is associated with the motion of the center of mass. For a closed system the center of mass moves with constant velocity. A change of momentum can only be caused by external forces. The total change of momentum caused by a force acting during a certain amount of time is called impulse. The internal interactions of a system of particles may be viewed as collision. In extreme cases, the particles may break into pieces or stick together after a collision. Such collisions are called inelastic. In the other extreme, the internal properties of the colliding particles remain unchanged, and the particles just exchange kinetic energy. Such collisions, in which the total kinetic energy is conserved, are called elastic.

5.1 Momentum and its Conservation

First, let us consider a single particle of mass m and velocity $\vec{v}$. Its momentum is then defined as

$$\vec{p} = m\vec{v}. \tag{5.1.1}$$

Newton's second law may then be written as

$$\frac{d\vec{p}}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a} = \vec{F}. \tag{5.1.2}$$

An external force acting on the particle changes its momentum. If there are no external forces we have

$$\frac{d\vec{p}}{dt} = 0, \tag{5.1.3}$$

i.e. momentum does not change with time — it is conserved. This is nothing but Newton's first law (in disguise).

Now, let us consider a system of N particles. Particle number i has mass m_i and its position (which may, of course, be time-dependent) is denoted by $\vec{r}_i$. Then each particle obeys Newton's law

$$m_i\vec{a}_i = m_i\frac{d^2\vec{r}_i}{dt^2} = \vec{F}_i, \tag{5.1.4}$$

where $\vec{F}_i$ is the total force acting on particle number i. This force is a sum of internal forces exerted by the other particles of the system and the external forces acting on the system of particles from outside. We denote the internal force acting on particle number i and exerted on it by particle number j by $\vec{F}_{ij}^{int}$. Then Newton's third law implies

$$\vec{F}_{ji}^{int} = -\vec{F}_{ij}^{int}. \tag{5.1.5}$$

The external force acting on particle i is denoted by $\vec{F}_i^{ext}$. Now the total force acting on particle i is given by

$$\vec{F}_i = \sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{ij}^{int} + \vec{F}_i^{ext}. \tag{5.1.6}$$

We have summed up the internal forces caused by the other particles with $j \neq i$. Now let us consider the total momentum of the system of particles which is defined as

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i\vec{v}_i. \tag{5.1.7}$$

How does the total momentum change with time? We find

$$\begin{aligned}
\frac{d\vec{P}}{dt} &= \sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} = \sum_{i=1}^{N} \vec{F}_i \\
&= \sum_{i=1}^{N} \left(\sum_{\substack{j=1 \\ j\neq i}}^{N} \vec{F}_{ij}^{int} + \vec{F}_i^{ext} \right) = \sum_{i=1}^{N} \vec{F}_i^{ext} = \vec{F}^{ext}.
\end{aligned} \tag{5.1.8}$$

The sum over the internal forces cancels because of Newton's third law. What remains is the total external force. If we have a closed system of particles (i.e. one without external forces acting on it) we have

$$\frac{d\vec{P}}{dt} = 0, \tag{5.1.9}$$

such that the total momentum is conserved. Of course, the momenta of individual particles will in general not be conserved, only their total sum is.

5.2 The Center of Mass

Let us read the equation

$$\frac{d\vec{P}}{dt} = \vec{F}^{ext} \tag{5.2.1}$$

as Newton's second law for some hypothetical "particle" of mass

$$M = \sum_{i=1}^{N} m_i, \tag{5.2.2}$$

and position $\vec{R}$ and velocity $\vec{V}$, such that

$$\vec{P} = M\vec{V} = M\frac{d\vec{R}}{dt}. \tag{5.2.3}$$

We have

$$\vec{P} = \sum_{i=1}^{N} \vec{p}_i = \sum_{i=1}^{N} m_i \vec{v}_i = \sum_{i=1}^{N} m_i \frac{d\vec{r}_i}{dt} = \frac{d}{dt} \sum_{i=1}^{N} m_i \vec{r}_i, \tag{5.2.4}$$

such that we can identify

$$\vec{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i = \frac{\sum_{i=1}^{N} m_i \vec{r}_i}{\sum_{i=1}^{N} m_i}. \tag{5.2.5}$$

The vector $\vec{R}$ determines the position of the center of mass of the system of particles. The system responds to external forces $\vec{F}^{ext}$ as if its total mass M was concentrated in the point $\vec{R}$.

Often it is useful to describe a system of particles in their center of mass frame. This is a reference frame in which the center of mass is at rest, i.e.

$$\vec{V} = \frac{d\vec{R}}{dt} = 0 \ \Rightarrow \ \vec{P} = M\vec{V} = 0. \tag{5.2.6}$$

In the center of mass frame the total momentum vanishes.

5.3 Inelastic and Elastic Collisions

Let us consider a system of two particles in their center of mass frame. Then their total momentum is given by

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = 0 \;\Rightarrow\; \vec{v}_2 = -\frac{m_1}{m_2}\vec{v}_1. \tag{5.3.1}$$

When the two particles interact with each other (for example, they may collide) the total momentum remains unchanged. After the collision the particles may, for example, break into pieces, such that we end up with an N particle system. Then we still have

$$\vec{P} = \sum_{i=1}^{N}\vec{p}_i = \sum_{i=1}^{N} m_i\vec{v}_i = 0. \tag{5.3.2}$$

In another extreme situation the two particles may stick together, such that after the collision we end up with a single particle. In that case $\vec{P} = 0$ implies that this particle will be at rest. Cases like these, in which particles change their internal structure, are called inelastic collisions. Let us consider the kinetic energy of our two particle system before the collision

$$E_{kin} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2. \tag{5.3.3}$$

This is certainly a positive number. After the collision, when the two particles stick together and come to rest, the kinetic energy is

$$E'_{kin} = 0, \tag{5.3.4}$$

i.e. the original kinetic energy must have been transformed into other forms of energy (e.g. heat). Whenever this happens we speak of inelastic collisions.

In general, collisions of particles are inelastic. There are, however, special cases in which particles do not change their internal structure. In these cases, not only the total energy, but even the kinetic energy alone is conserved. Collisions in which kinetic energy is conserved are called elastic. Let us consider an elastic collision of two particles in their center of mass frame. Before the collision we have

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 = 0, \;\; E_{kin} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2, \tag{5.3.5}$$

and after the collision

$$\vec{P} = m_1\vec{v}\,'_1 + m_2\vec{v}\,'_2 = 0, \;\; E_{kin} = \frac{1}{2}m_1 {v'_1}^2 + \frac{1}{2}m_2 {v'_2}^2. \tag{5.3.6}$$

Because we are in the center of mass frame, we have

$$\vec{v}_2 = -\frac{m_1}{m_2}\vec{v}_1, \ \ \vec{v}_2' = -\frac{m_1}{m_2}\vec{v}_1', \tag{5.3.7}$$

and hence

$$E_{kin} = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2\frac{m_1^2}{m_2^2}v_1^2 = \frac{1}{2}m_1 {v_1'}^2 + \frac{1}{2}m_2\frac{m_1^2}{m_2^2}{v_1'}^2, \tag{5.3.8}$$

such that $v_1^2 = {v_1'}^2$ and $v_2^2 = {v_2'}^2$, i.e. the speeds of the particles remain the same, but their direction of motion will in general change.

5.4 Impulse

As we have seen, momentum is conserved when no external forces act on a particle or on a system of particles. When a force acts, however, momentum changes because

$$\frac{d\vec{p}(t)}{dt} = \vec{F}(t). \tag{5.4.1}$$

One uses the concept of impulse to describe the change of momentum due to some force $\vec{F}$ acting during a certain time interval. Impulse is defined as

$$\vec{I} = \int_{t_1}^{t_2} dt\ \vec{F}(t) = \int_{t_1}^{t_2} dt\ \frac{d\vec{p}(t)}{dt} = \vec{p}(t_2) - \vec{p}(t_1). \tag{5.4.2}$$

Impulse is a concept analogous to work, which describes the change in kinetic or potential energy. Now, however, we integrate the force over time instead of distance. Impulse is a particularly useful concept when forces act only during a very short time, like e.g. the contact forces in a collision. Then it may be impossible to understand in detail how momentum changes as a function of time, but still we may know the net change of momentum.

Chapter 6

Dissipative Forces

As we know, energy is conserved. This does, however, not imply that mechanical energy (kinetic or potential) is always conserved. There are other non-mechanical forms of energy like heat, chemical energy, or radiation. For example, some forces relevant in classical mechanics turn mechanical energy into heat. Such forces are known as dissipative forces. The most important dissipative force is friction. On a microscopic level friction is due to fundamental electromagnetic interactions. On a macroscopic level, it is, however, much more practical to describe friction forces by some effective force laws.

6.1 Friction

Let us consider a body that is at rest on some surface and friction is not completely negligible. When we exert a small force F on the body, static friction prevents it from accelerating in the direction of F. In other words, there is a static friction force $\mathcal{F}_s$ that exactly compensates the force F. Of course, when F becomes larger and larger, the massive body will eventually start accelerating. This is the case when F exceeds the maximum friction force. This maximal force is proportional to the normal force N and is given by

$$\mathcal{F}_{s,max} = \mu_s N, \ \mathcal{F}_s \leq \mathcal{F}_{s,max}, \tag{6.1.1}$$

where μ_s is the coefficient of static friction. Once the body starts moving, the friction force changes — it turns into kinetic friction — which is directed opposite to the direction of motion and given by

$$\mathcal{F}_k = \mu_k N, \tag{6.1.2}$$

where μ_k is the coefficient of static friction. One has $\mu_k < \mu_s$, i.e. it is easier to keep moving than to initiate motion.

Chapter 7

Rotation about an Axis

In this chapter we introduce the concept of a rigid body. This is an idealized body that is not deformable. Indeed, many real solids are well approximated by perfectly rigid bodies. Compared to a point particle, an extended rigid body has more degrees of freedom. Like a point particle, a rigid body can undergo translational motion. However, in addition it can also rotate about an arbitrary axis. In this chapter we restrict ourselves to rotations in two dimensions.

7.1 Rotation of a Rigid Body

Let us consider a flat rigid body moving in the plane. Such a motion can be decomposed into translational and rotational components. For simplicity, let us consider rotation about some fixed axis perpendicular to the plane. The rotation can be described by some angle θ which is a function of time. Then it is useful to introduce the angular velocity

$$\omega = \frac{d\theta}{dt}, \tag{7.1.1}$$

and the angular acceleration

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}. \tag{7.1.2}$$

What is the kinetic energy of the rigid body? We must sum up the kinetic

quantity	translation	rotation	quantity
position	$x(t)$	$\theta(t)$	angle
velocity	$v = \frac{dx}{dt}$	$\omega = \frac{d\theta}{dt}$	angular velocity
acceleration	$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$	$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$	angular acceleration
mass	$M = \sum_i m_i$	$I = \sum_i m_i r_i^2$	moment of inertia
kinetic energy	$E_{kin} = \frac{1}{2} M v^2$	$E_{kin} = \frac{1}{2} I \omega^2$	kinetic energy
momentum	$P = Mv$	$L = I\omega$	angular momentum
force	$F = Ma$	$\tau = I\alpha$	torque

Table 7.1: *Analogies between physical quantities for translational and rotational motion.*

energies of all its pieces

$$E_{kin} = \sum_i \frac{1}{2} m_i v_i^2 = \sum_i \frac{1}{2} m_i (r_i \omega)^2 = \sum_i \frac{1}{2} m_i r_i^2 \omega^2 = \frac{1}{2} I \omega^2. \tag{7.1.3}$$

Here r_i measures the distance of the point particle number i of mass m_i from the rotation axis. We have introduced the moment of inertia

$$I = \sum_i m_i r_i^2, \tag{7.1.4}$$

which does depend on the axis about which the rigid body is rotating. The moment of inertia measures how difficult it is to speed up the rotation of the rigid body.

Another important concept (analogous to momentum for translational motion) is angular momentum

$$L = I\omega. \tag{7.1.5}$$

If there are no external forces, (just like momentum) angular momentum is conserved. Table 7.1 compares translational with rotational motion The analog of force is torque. If a force $\vec{F}_i$ acts on the mass m_i inside the rigid body only its component $F_{i\perp}$ can speed up the rotational motion

$$F_{i\perp} = F_i \sin \varphi_i. \tag{7.1.6}$$

Here φ_i is the angle between the force $\vec{F}_i$ and the position vector $\vec{r}_i$ of the particle. Next, we use Newton's equation

$$F_{i\perp} = m_i a_{i\perp} \tag{7.1.7}$$

together with the expression for the perpendicular acceleration

$$a_{i\perp} = r_i \alpha, \tag{7.1.8}$$

such that

$$F_i \sin\varphi_i = F_{i\perp} = m_i r_i \alpha. \tag{7.1.9}$$

We multiply with r_i and we sum over all particles i in the rigid body

$$\sum_i F_i r_i \sin\varphi_i = \sum_i m_i r_i^2 \alpha = I\alpha. \tag{7.1.10}$$

Now we can identify the total torque

$$\tau = \sum_i F_i r_i \sin\varphi = \sum_i \tau_i, \ \ \tau_i = F_i r_i \sin\varphi_i. \tag{7.1.11}$$

Just as all internal forces in a system of particles cancel, as a consequence of Newton's third law ($\vec{F}_{21} = -\vec{F}_{12}$) also the internal torques cancel. This follows because for two particles 1 and 2 inside the rigid body we have $r_1 \sin\varphi_1 = r_2 \sin\varphi_2$ such that $\tau_2 = -\tau_1$.

7.2 Parallel and Perpendicular Axes Theorems

Let us compute the moments of inertia for rotations of a rigid body about two parallel axes, one going through the center of mass. The moment of inertia for rotations about the axis through the center of mass is

$$I_{cm} = \sum_i m_i r_i^2. \tag{7.2.1}$$

Now we consider another axis displaced from the center of mass by a vector $\vec{d}$

$$\begin{aligned} I &= \sum_i (\vec{d} - \vec{r}_i)^2 = \sum_i m_i (d^2 - 2\vec{d}\cdot\vec{r}_i + r_i^2) \\ &= M d^2 - 2\vec{d}\cdot\sum_i m_i \vec{r}_i + \sum_i m_i r_i^2. \end{aligned} \tag{7.2.2}$$

We did this calculation in the center of mass frame with the center of mass at the origin, such that

$$\vec{R} = \frac{1}{M}\sum_i m_i \vec{r}_i = 0. \tag{7.2.3}$$

Then we obtain the parallel axes theorem

$$I = Md^2 + I_{cm}. \tag{7.2.4}$$

The moment of inertia about a general axis is always larger than the one about an axis through the center of mass. The difference between the two is Md^2 — the mass of the rigid body times the distance between the two axes squared.

Next, let us consider rotations about axes perpendicular to each other, but through the same point (not necessarily the center of mass). Then we have

$$I_x = \sum_i m_i y_i^2, \; I_y = \sum_i m_i x_i^2, \; I_z = \sum_i (x_i^2 + y_i^2) = I_x + I_y. \tag{7.2.5}$$

This is the so-called perpendicular axes theorem. It applies only to bodies which are absolutely flat and which lie in the x-y-plane.

Chapter 8

Rotations in Three Dimensions

General rotations are not limited to two dimensions. Instead, they are 3-dimensional. Then quantities like torque and angular momentum turn out to be vectors. To deal with them, we need a new mathematical concept — the vector cross product of two vectors. In this chapter we will also discuss the conditions for equilibrium of rigid bodies. Then all external forces and all external torques must add up to zero.

8.1 Torque as a Vector

For the moment, let us consider 2-dimensional motion again. The torque due to the force F is

$$\tau = rF\sin\varphi, \tag{8.1.1}$$

which we identify as the length of the vector $\vec{r} \times \vec{F}$. In the general 3-dimensional case we define the torque as

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{8.1.2}$$

It should be noted that this definition refers to a particular choice of origin of the coordinate system.

In the special case of two dimensions we have $z = 0$ and $F_z = 0$ such that

$$\begin{aligned} \vec{\tau} &= \vec{r} \times \vec{F} = (yF_z - zF_y, zF_x - xF_z, xF_y - yF_x) \\ &= (0, 0, xF_y - yF_x) = (0, 0, \tau_z). \end{aligned} \tag{8.1.3}$$

In other words, in two dimensions we were dealing with the z-component of the torque vector, while its x- and y-components were equal to zero. Of course, in a general 3-dimensional situation this is no longer the case.

8.2 Angular Momentum

So far we have talked about angular momentum only for 2-dimensional rotations of a rigid body ($L = I\omega$). However, angular momentum is a much more general concept. In fact, it is as important as energy or momentum. In particular, angular momentum is conserved. Let us first consider a single particle at position $\vec{r}$ with momentum $\vec{p}$. Then its angular momentum is defined as

$$\vec{L} = \vec{r} \times \vec{p}. \tag{8.2.1}$$

Let us calculate the change of angular momentum with time

$$\begin{aligned} \frac{d\vec{L}}{dt} &= \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \vec{v} \times m\vec{v} + \vec{r} \times m\frac{d\vec{v}}{dt} \\ &= m\vec{v} \times \vec{v} + \vec{r} \times m\vec{a} = \vec{r} \times \vec{F} = \vec{\tau}. \end{aligned} \tag{8.2.2}$$

Here we have used $\vec{v} \times \vec{v} = 0$. We conclude that angular momentum changes only if some torque is applied. In the absence of torque

$$\frac{d\vec{L}}{dt} = 0, \tag{8.2.3}$$

i.e. angular momentum is conserved.

Angular momentum conservation is related to the fact that the laws of Nature are the same if we describe them in a reference frame that is rotated by any fixed angle with respect to the original reference frame. In particular, the gravitational force (and any other central force) is of that kind. Let us consider the earth orbiting around the sun. Then the force exerted on the earth is

$$\vec{F} = -\frac{GM_sM_e}{r^3}\vec{r}, \tag{8.2.4}$$

i.e. it is anti-parallel to $\vec{r}$ such that the torque on the earth is

$$\vec{\tau} = \vec{r} \times \vec{F} = 0. \tag{8.2.5}$$

Hence, the angular momentum of the earth is, in fact, conserved.

Now let us consider the angular momentum of a rigid body rotating in three dimensions about some axis with direction $\vec{e}_\omega$ with angular velocity ω. The rotation can then be described by the angular velocity vector $\vec{\omega} = \omega\vec{e}_\omega$. For a particular part of the body at position $\vec{r}_i$, the velocity vector is then given by

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i. \tag{8.2.6}$$

Let us now calculate the total angular momentum of the rigid body by summing over all its pieces

$$\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i = \sum_i m_i\vec{r}_i \times \vec{v}_i = \sum_i m_i\vec{r}_i \times (\vec{\omega} \times \vec{r}_i). \tag{8.2.7}$$

This is the 3-dimensional analog of $L = I\omega$. In the 2-dimensional case we have

$$\vec{\omega} = \omega\vec{e}_z, \ \vec{r}_i = x_i\vec{e}_x + y_i\vec{e}_y, \tag{8.2.8}$$

such that

$$\vec{\omega} \times \vec{r}_i = \omega\vec{e}_z \times (x_i\vec{e}_x + y_i\vec{e}_y) = \omega(x_i\vec{e}_y - y_i\vec{e}_x), \tag{8.2.9}$$

and hence

$$\begin{aligned} \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) &= (x_i\vec{e}_x + y_i\vec{e}_y) \times \omega(x_i\vec{e}_y - y_i\vec{e}_x) \\ &= \omega(x_i^2\vec{e}_z + y_i^2\vec{e}_z) = \omega r_i^2\vec{e}_z. \end{aligned} \tag{8.2.10}$$

Therefore, for 2-dimensional rotations we have

$$\vec{L} = \sum_i m_i\omega r_i^2\vec{e}_z = \sum_i m_i r_i^2\omega\vec{e}_z \ \Rightarrow \ L_z = I\omega, \tag{8.2.11}$$

with $I = \sum_i m_i r_i^2$. In this case $\vec{L}$ and $\vec{\omega}$ are parallel. This is, however, not the case for general 3-dimensional rotations. For example, the precession of gyroscopes arises because in that case $\vec{L}$ and $\vec{\omega}$ are not parallel. The axis $\vec{\omega}$ of the gyroscope then rotates about the conserved $\vec{L}$ vector.

8.3 Statics

In the subject of statics we ask if a rigid body (or a system of rigid bodies) is in equilibrium. This is the case if all external forces and all external torques add up to zero

$$\vec{F} = \sum_i \vec{F}_i = 0, \ \vec{\tau} = \sum_i \vec{\tau}_i = 0. \tag{8.3.1}$$

If this were not the case, the body would accelerate or start rotating. Let us convince ourselves that the arbitrary choice of the origin of the coordinate system does not influence the above conditions for equilibrium, although the values of the individual torques in general depend on this. Let us calculate the total torque in another reference frame with the origin shifted by a vector $\vec{d}$. We then have

$$\vec{\tau}' = \sum_i \vec{\tau}_i' = \sum_i \vec{r}_i' \times \vec{F}_i = \sum_i (\vec{r}_i - \vec{d}) \times \vec{F}_i = \sum_i \vec{r}_i \times \vec{F}_i - \vec{d} \times \sum_i \vec{F}_i = \vec{\tau}. \quad (8.3.2)$$

Here we have used $\sum_i \vec{F}_i = 0$. Since (in equilibrium) the total torque is independent of the choice of reference frame, we can make any choice we want. Sometimes, if one makes a clever choice, the calculations simplify.

Chapter 9

Kinetic Theory of the Ideal Gas

As an introduction to the subject of thermodynamics, let us study gases consisting of weakly interacting atoms or molecules. The atoms or molecules of a gas move more or less independent of each other. One can use an idealization to describe this situation: in an ideal gas, the atoms or molecules move independently as free particles, except during collisions which are assumed to be completely elastic. In practice it is completely impossible (and fortunately unnecessary) to describe all the degrees of freedom (of order 10^{23}) in detail. It is much more practical to describe the gas particles from a statistical point of view. Their average force on the walls of a container determines the pressure of the gas, and the average kinetic energy of the particles determines the temperature. Pressure and temperature are directly measurable physical quantities which characterize the gas much better than a list of all positions and momenta of the gas particles. Pressure, temperature, and density of a gas are related to each other by the ideal gas law. When a gas is heated, it increases its internal energy (and hence its temperature) and it may also expand and thereby do work. The energy balance of a gas is summarized in the first law of thermodynamics which reflects nothing but energy conservation. To get acquainted with gases, we will first consider their elementary constituents: atoms and molecules, and study some of their properties. Some properties of a gas depend on if the gas particles are atoms or molecules (monatomic versus diatomic gases).

9.1 Atoms and Molecules

Atoms consist of an atomic nucleus and a number of electrons. The nucleus consists of Z positively charged protons and N neutrons and has a size of the order of 10^{-14} m. The whole atom is electrically neutral because there are also Z negatively charged electrons forming a cloud surrounding the nucleus. The size of the electron cloud (and of the entire atom) is of the order of 10^{-10} m. The mass of a neutron or proton is one atomic mass unit $M_n \approx M_p = 1u = 1.66 \times 10^{-24}$ g, while the electron mass is much smaller ($M_e = 9.04 \times 10^{-28}$ g). Hence, the mass of the atom is almost entirely concentrated in the nucleus and we can write it as $M_A = (Z + N)u$.

To understand the physics of the electron cloud as well as of the nucleus we need quantum mechanics. However, at moderate temperatures the energy is insufficient to ionize the atoms and we can treat the atoms as point-like and structureless. This is exactly what we do for a monatomic ideal gas. Of course, this is an idealization. In particular, at very high temperatures the electrons could be removed from the atomic nucleus and the system becomes a plasma (as e.g. the matter that exists in the sun). At yet much higher temperatures even the atomic nucleus itself would dissolve into quarks and gluons and we would end up in a quark-gluon plasma — the state of matter that existed during the first microsecond after the big bang.

The simplest atom is the hydrogen atom H. It consists of one proton (the nucleus) and one electron in the cloud and has a mass $M_H = 1u$. The electrons are bound to the atomic nucleus by electromagnetic forces. These forces also bind atoms to molecules. For example, two hydrogen atoms may form a hydrogen molecule H_2, by sharing their electron cloud. The diatomic hydrogen molecule has a mass $M_{H_2} = 2u$. Table 9.1 summarizes some properties of atoms and molecules.

How many water molecules are contained in 1 cm^3 of water? We know that 1 liter $= 10^3$ cm^3 of water weighs 1 kg. Hence, 1 cm^3 of water weighs 1 g. One water molecule weighs $18u$, so the number of water molecules in 1 cm^3 of water is

$$N = \frac{1\text{g}}{18u} = \frac{1\text{g}}{2.98 \times 10^{-23}\text{g}} = 3.36 \times 10^{22}. \tag{9.1.1}$$

Consequently, 18 g of water contain Avogadro's number

$$N_A = 18N = 6.02 \times 10^{23} \tag{9.1.2}$$

of water molecules. An amount of matter that contains N_A basic units (atoms

Particle	notation	Z	N	M_A
hydrogen atom	H	1	0	$1u$
helium atom	He	2	2	$4u$
nitrogen atom	N	7	7	$14u$
oxygen atom	O	8	8	$16u$
hydrogen molecule	H_2	2	0	$2u$
nitrogen molecule	N_2	14	14	$28u$
oxygen molecule	O_2	16	16	$32u$
water molecule	H_2O	10	8	$18u$

Table 9.1: *Basic Properties of some atoms and molecules.*

or molecules) of some substance is called one mole of that substance. One mole of water weighs 18 g, and in general one mole of a substance weighs the number of its basic units in grams. For example, one mole of oxygen gas (O_2) weighs 32 g, while one mole of helium gas (He) weighs 4 g. The first contains N_A oxygen molecules, the second N_A helium atoms.

9.2 Pressure and Temperature of an Ideal Gas

Let us consider a container of volume $V = L_x \times L_y \times L_z$ containing an ideal gas consisting of N particles (atoms or molecules) of mass M. The number density of gas particles is then given by $n = N/V$ and the mass density is $\rho = NM/V = Mn$. The gas particles perform a random, chaotic motion, and each has its own velocity $\vec{v}_i$. The particles collide with each other and with the walls of the container. For an ideal gas we assume that all these collisions are completely elastic.

Let us consider the average force that the gas exerts on the walls of the container. This will lead to an expression for the pressure of the gas. When particle number i collides with the wall (perpendicular to the x-direction) its velocity $\vec{v}_i$ changes to $\vec{v}_i'$. Since the collision is elastic we have $v_{ix}' = -v_{ix}$. Hence, during the collision particle i transfers an impulse $I_{ix} = 2Mv_{ix}$ to the wall (in the perpendicular x-direction). What is the probability for particle i hitting the wall during a time interval Δt? In order to be able to reach the wall within Δt, the particle must at most be a distance $\Delta x_i = v_{ix}\Delta t$ away from the wall. Since the wall has area $A = L_y \times L_z$, it must be inside a volume $A\Delta x_i$. Since the total volume is V, the probability to be within the volume $A\Delta x_i$ is $A\Delta x_i/V$. Still, the particle will not necessarily hit the wall, even if it is within the volume $A\Delta x_i$.

In half of the cases it will move away from the wall. Hence, the probability for particle i to hit the wall during the time Δt is only

$$\frac{1}{2}\frac{A\Delta x_i}{V} = \frac{1}{2}\frac{Av_{ix}\Delta t}{V}. \tag{9.2.1}$$

The force (impulse per time) exerted on the wall by particle i is hence given by

$$F_{ix} = \frac{2Mv_{ix}}{\Delta t}\frac{1}{2}\frac{Av_{ix}\Delta t}{V} = \frac{A}{V}Mv_{ix}^2, \tag{9.2.2}$$

and the total force exerted by all particles is

$$F_x = \sum_{i=1}^{N} F_{ix} = \frac{A}{V}\sum_{i=1}^{N} Mv_{ix}^2 = \frac{A}{V}N\langle Mv_x^2\rangle. \tag{9.2.3}$$

We have introduced the average over all particles

$$\langle Mv_x^2\rangle = \frac{1}{N}\sum_{i=1}^{N} Mv_{ix}^2. \tag{9.2.4}$$

The force (perpendicular to the wall) per unit area of the wall is the pressure which is hence given by

$$p = \frac{F_x}{A} = \frac{N}{V}\langle Mv_x^2\rangle = \frac{N}{V}\frac{2}{3}\langle\frac{M}{2}v^2\rangle. \tag{9.2.5}$$

Here we have introduced the velocity squared $v^2 = v_x^2 + v_y^2 + v_z^2$ and we have used symmetry to argue that $\langle v_x^2\rangle = \langle v_y^2\rangle = \langle v_z^2\rangle$. Now we can write

$$pV = N\frac{2}{3}\langle\frac{M}{2}v^2\rangle. \tag{9.2.6}$$

In other words, the pressure of the gas is proportional to the average kinetic energy of the gas particles $\langle\frac{1}{2}Mv^2\rangle$. Pressure is measured in Pascal (1 Pa = 1 N/m^2) and the typical atmospheric pressure is about 10^5 Pa.

The absolute temperature T of the gas is defined by

$$\frac{3}{2}k_B T = \langle\frac{M}{2}v^2\rangle. \tag{9.2.7}$$

Up to the numerical factor $\frac{3}{2}k_B$ the temperature is just the average kinetic energy of the gas particles. The Boltzmann constant k_B is present to match the different units of temperature and energy. If we (or Boltzmann and his colleagues) had decided to measure temperature in Joules (J), k_B could have been dropped. However, temperature is traditionally measured in degrees Kelvin (K) and

$$k_B = 1.38 \times 10^{-23} \text{J/K}. \tag{9.2.8}$$

From its definition it is clear that the absolute temperature must be positive, i.e. $T \geq 0$, because $v^2 \geq 0$. Only if all gas particles are at rest we have $T = 0$. This corresponds to the absolute zero of temperature (0 K). In degrees Celsius this corresponds to -273.16 C.

With the above definition of temperature we now obtain

$$pV = Nk_BT. \tag{9.2.9}$$

This is the ideal gas law. Sometimes it is also written as

$$pV = \mathcal{N}RT, \tag{9.2.10}$$

where $\mathcal{N} = N/N_A$ is the number of moles of gas, and $R = k_BN_A = 8.3$ J/K is the so-called gas constant.

A monatomic ideal gas has no internal degrees of freedom. In contrast to diatomic gases, the particles in a monatomic ideal gas are considered as point-like and cannot rotate or vibrate. The average energy $\langle E \rangle$ of a monatomic ideal gas is hence just its kinetic energy, i.e.

$$\langle E \rangle = N \langle \frac{M}{2} v^2 \rangle = \frac{3}{2} Nk_BT, \tag{9.2.11}$$

and we can also write the ideal gas law as

$$pV = \frac{2}{3} \langle E \rangle. \tag{9.2.12}$$

9.3 Internal Energy of Monatomic and Diatomic Gases

Let us consider a monatomic ideal gas. Its internal energy U is nothing but the total kinetic energy of the gas atoms

$$U = N \langle \frac{1}{2} mv^2 \rangle = \frac{3}{2} Nk_BT \;\Rightarrow\; pV = \frac{2}{3} U. \tag{9.3.1}$$

As opposed to the point-like atoms of a monatomic gas, the molecules of a diatomic gas can also rotate. Hence, the total internal energy contains a term U_{rot} related to the rotational motion of molecules

$$U = N\langle\frac{1}{2}mv^2\rangle + U_{rot} = \frac{3}{2}Nk_BT + U_{rot} \Rightarrow$$
$$pV = \frac{2}{3}(U - U_{rot}) = \frac{2}{3}\left(1 - \frac{U_{rot}}{U}\right)U = (\gamma - 1)U, \qquad (9.3.2)$$

where we have defined

$$\gamma = 1 + \frac{2}{3}\left(1 - \frac{U_{rot}}{U}\right). \qquad (9.3.3)$$

For a monatomic gas there is no rotation ($U_{rot} = 0$) and hence

$$\gamma = 1 + \frac{2}{3} = \frac{5}{3}. \qquad (9.3.4)$$

A diatomic molecule can rotate about two axes perpendicular to the line connecting the two atoms. It cannot rotate about an axis through the two atoms, because we may consider the atoms as point-like. A diatomic molecule hence has two rotational degrees of freedom. In addition, it has three translational degrees of freedom. This leads to a naive estimate

$$\frac{U_{rot}}{U} = \frac{2}{2+3} = \frac{2}{5}, \qquad (9.3.5)$$

which implies

$$\gamma = 1 + \frac{2}{3}\left(1 - \frac{2}{5}\right) = 1 + \frac{2}{3} \times \frac{3}{5} = \frac{7}{5}. \qquad (9.3.6)$$

This number is close to the actual value of γ for diatomic gases. Still, the above estimate is very naive and a true understanding of the value of γ requires the use of quantum mechanics.

9.4 Heat Transfer and Energy Conservation

One can change the internal energy of a gas by transferring heat to it. This will increase the temperature of the gas. Alternatively, it can also lead to an expansion of the gas, which implies that the gas is doing work. Let us consider a gas in a container closed by a frictionless piston. The air outside the container exerts a constant pressure p on the piston. The piston comes to an equilibrium position when the pressure of the gas also equals p. When the gas is heated, it expands and pushes the piston further out by a distance dx, against the force

$F = pA$ of the external air. Here A is the area of the piston. This means that the gas has done the work

$$dW = Fdx = pAdx = pdV, \tag{9.4.1}$$

where $dV = Adx$ is the volume by which the gas has expanded.

Not all of the heat transferred to the gas from outside will turn into work done by the gas. Part of it will lead to an increase of the internal energy (and hence of the temperature) of the gas. The energy balance of the gas takes the form

$$dU = dQ - dW, \tag{9.4.2}$$

where dU is the change of internal energy, dQ is the heat transferred to the gas, and dW is the work done by the gas. This is the first law of thermodynamics, which just represents energy conservation. Here a new form of energy — namely heat — enters the energy balance equation.

Let us consider the expansion of a gas without heat transfer ($dQ = 0$). This is called adiabatic expansion. From before we have

$$pV = (\gamma - 1)U \ \Rightarrow \ (\gamma - 1)dU = pdV + dpV. \tag{9.4.3}$$

From the energy balance equation we obtain

$$\begin{aligned} & dU = -dW = -pdV \ \Rightarrow \ -(\gamma - 1)pdV = pdV + dpV \ \Rightarrow \\ & -\gamma pdV = dpV \ \Rightarrow \ \frac{dp}{p} = -\gamma \frac{dV}{V}. \end{aligned} \tag{9.4.4}$$

We integrate this equation from initial to final pressures and volumes, i.e.

$$\int_{p_i}^{p_f} \frac{dp}{p} = -\gamma \int_{V_i}^{V_f} \frac{dV}{V}, \tag{9.4.5}$$

and we obtain

$$\log p_f - \log p_i = -\gamma(\log V_f - \log V_i) \ \Rightarrow \ \frac{p_f}{p_i} = \left(\frac{V_i}{V_f}\right)^\gamma. \tag{9.4.6}$$

Hence, for an adiabatic expansion the initial and final pressures and volumes are related by

$$p_f V_f^\gamma = p_i V_i^\gamma. \tag{9.4.7}$$

Using the ideal gas law

$$p_f V_f = N k_B T_f, \ p_i V_i = N k_B T_i, \tag{9.4.8}$$

we obtain

$$\frac{T_f}{T_i} = \frac{p_f V_f}{p_i V_i} = \left(\frac{V_i}{V_f}\right)^{\gamma-1}. \tag{9.4.9}$$

In all cases $\gamma > 1$. Hence, expansion (i.e. $V_f > V_i$) implies cooling (i.e. $T_f < T_i$), at least if no heat is transferred to the gas from outside.

Now we will consider expansion without cooling. This is possible only if heat is transferred from outside. We want to supply enough energy to keep the temperature constant. Such a process is called isothermal expansion. In order to maintain a constant temperature, the internal energy must remain unchanged ($dU = 0$). The energy balance equation hence implies

$$dU = dQ - dW = 0 \;\Rightarrow\; dQ = dW = pdV = Nk_BT\frac{dV}{V}. \tag{9.4.10}$$

The total work done by the gas then equals the heat transferred to the gas and is given by

$$W = Nk_BT\int_{V_i}^{V_f}\frac{dV}{V} = Nk_BT(\log V_f - \log V_i) = Nk_BT\log\frac{V_f}{V_i}. \tag{9.4.11}$$

9.5 Molar Heat Capacities

How much heat must be transferred to a gas in order to increase its temperature by a certain amount? From our previous considerations it is clear that this depends on if the gas expands (i.e. does work) at the same time. First, we consider heating the gas at constant volume (i.e. without expansion). The heat required to increase the temperature of one mole of gas by one degree at constant volume is called the molar heat capacity C_V. Without expansion we have $dW = pdV = 0$ and hence $dU = dQ$ such that for a monatomic gas ($U = \frac{3}{2}Nk_BT$)

$$C_V = \frac{dQ}{\mathcal{N}dT} = \frac{dU}{\mathcal{N}dT} = \frac{3}{2}\frac{Nk_B}{\mathcal{N}} = \frac{3}{2}N_Ak_B = \frac{3}{2}R. \tag{9.5.1}$$

Now let us assume that the gas expands while heating, but that the pressure is constant. Then we have a corresponding quantity

$$C_p = \frac{dQ}{\mathcal{N}dT}, \tag{9.5.2}$$

but now

$$dU = dQ - dW = dQ - pdV \;\Rightarrow\; dQ = dU + pdV, \tag{9.5.3}$$

such that

$$C_p = \frac{dU}{\mathcal{N} dT} + \frac{p dV}{\mathcal{N} dT}. \tag{9.5.4}$$

We now use the ideal gas law $pV = Nk_BT$ to arrive at

$$\frac{dV}{dT} = \frac{Nk_B}{p}, \tag{9.5.5}$$

Taking the result for C_V of a monatomic gas we obtain

$$C_p = \frac{3}{2}R + \frac{Nk_B}{\mathcal{N}} = \frac{3}{2}R + N_A k_B = \frac{3}{2}R + R = \frac{5}{2}R. \tag{9.5.6}$$

The ratio of the molar heat capacities for a monatomic gas is $C_p/C_V = 5/3$. In general, this ratio is

$$\frac{C_p}{C_V} = \gamma. \tag{9.5.7}$$

Chapter 10

Fluid Mechanics

In an ideal gas the atoms or molecules move independent of each other, except that they may collide elastically. In real gases, however, particles are not completely independent because they interact with each other. For example, two water molecules H_2O in steam exert a certain force on each other, which depends on their distance r. The force may be derived from a potential energy function $V(r)$. At large distances the potential energy goes to zero, and hence the two molecules no longer interact. At very short distances the energy becomes large leading to a repulsive force between the two molecules. At intermediate distances the potential energy has a minimum value $V(r_0) = -V_0$ which corresponds to a preferred distance r_0 between the two molecules. At high temperatures $k_B T \gg V_0$ the average kinetic energy is so large that the potential energy can be neglected. Then, for all practical purposes, we have an almost ideal gas. As the temperature is lowered to $k_B T \approx V_0$, the average kinetic energy decreases and the potential energy becomes more important, i.e. the molecules now are more often at their preferred distance r_0. At about this temperature the properties of the system of molecules change drastically. At a critical temperature a phase transition occurs, at which the gas turns into a liquid. In the liquid phase, the molecules are strongly correlated (they are more or less at the distance r_0) but still they perform a random chaotic motion, unlike in a solid with a regular crystal lattice.

Since in a liquid the molecules are more or less at the fixed distance r_0, the density of the liquid is more or less constant. We say that liquids of constant density are incompressible. Hence, unlike a gas, a liquid will not expand and fill its container completely. In a liquid the molecules are strongly correlated. On macroscopic scales this leads to internal friction effects and also to friction with the walls of the container. The friction in a liquid is characterized by its viscosity.

As for gases, in order to simplify the theoretical analysis, we want to make some idealization. Hence, we introduce the concept of an ideal fluid which has zero viscosity and is completely incompressible, i.e. it has a constant density ρ. Just as a gas, also a fluid exerts a certain pressure on the wall of its container. This pressure is given by Pascal's law to which we now turn.

10.1 Pascal's Law and Archimedes' Principle

Let us consider the pressure p in a fluid as a function of the depth below the surface. We consider a volume $dV = Adz$ of liquid. The forces acting on it are gravity $-Mg = -\rho dVg$ and pressure forces from above $-p(z)A$ and from below $p(z-dz)A$. Equilibrium of forces implies

$$\begin{aligned} & -Mg - p(z)A + p(z-dz)A = 0 \ \Rightarrow \\ & p(z) - p(z-dz) = -\frac{Mg}{A} = -\frac{\rho dVg}{A} = -\rho g dz \ \Rightarrow \\ & \frac{dp(z)}{dz} = \lim_{dz\to 0} \frac{p(z)-p(z-dz)}{dz} = -\rho g. \end{aligned} \tag{10.1.1}$$

We integrate this equation from the surface at pressure $p(0)$ and we obtain

$$p(z) - p(0) = \int_0^z dz\ \frac{dp(z)}{dz} = -\int_0^z dz\ \rho g = -\rho g z. \tag{10.1.2}$$

This is Pascal's law.

Next, we consider Archimedes' principle. For this purpose, let us consider an arbitrarily shaped region in the liquid with volume V. Gravity exerts a force $-\rho gV$ on this volume of liquid acting on its center of mass. The external pressure forces exactly compensate the gravitational force. In other words, they add up to ρgV. Now, let us remove the liquid from the volume V and replace it by some other object of the same shape and mass M. The exerted pressure forces are unaffected by the replacement, and hence they still add up to ρgV (acting on the center of mass of the now removed water). Gravity now exerts a force $-Mg$ acting on the center of mass of the new object. The total force is $-Mg + \rho gV$. There may also be a net torque if the two centers of mass do not coincide.

If $Mg > \rho gV$ the mass of the object is larger than that of the displaced water and the object sinks (there is a net downward force). If $Mg < \rho gV$ the object rises to the surface. The condition for rising or sinking can also be expressed in terms of the average density

$$\rho_{ave} = \frac{M}{V} \tag{10.1.3}$$

of the object. If $\rho_{ave} > \rho$ the object sinks, and if $\rho_{ave} < \rho$ the object rises to the surface. Suppose $\rho_{ave} < \rho$, how much of the object would then be submerged in the liquid? Now the object displaces liquid of volume V' only. This liquid has a mass $\rho V'$. In order to balance gravity Mg on the object, we need

$$Mg = \rho V' g \;\Rightarrow\; \rho V' = M. \tag{10.1.4}$$

This is Archimedes' principle: the amount of liquid an object of mass M displaces also has mass M. Hence, the volume V' of the object submerged in the liquid is given by $V' = M/\rho$.

10.2 Continuity Equation and Bernoulli's Equation

A fluid at rest is characterized by the pressure $p(\vec{r})$ as a function of position $\vec{r}$ (and by its constant density). A fluid in motion, on the other hand, is described by its velocity $\vec{v}(\vec{r}, t)$ as a function of position $\vec{r}$ and time t. Also the pressure $p(\vec{r}, t)$ may now be time-dependent. The general motion of fluids may be very complex, in particular, chaotic, turbulent flow patterns may arise. In order to avoid such complications, here we restrict ourselves to steady (i.e. time-independent) flows, i.e. $p(\vec{r})$ and $\vec{v}(\vec{r})$ are functions of position only.

Let us consider the steady flow of an incompressible fluid through a tube of varying cross section A. If the fluid enters at A_1 with velocity v_1 (perpendicular to A_1) a volume $dV = A_1 v_1 dt$ enters the tube during the time dt. Since the fluid is incompressible, the same volume must leave the tube at the other end. This implies the so-called continuity equation

$$A_1 v_1 = A_2 v_2, \tag{10.2.1}$$

which just represents mass conservation.

What is the pressure of a liquid in steady motion? We again consider a tube of varying cross section. What is the work done by the pressure forces? We consider the volume $dV = A_1 v_1 dt$ that enters the tube at pressure p_1. The pressure does work $dW_1 = p_1 dV$. The same amount of liquid $dV = A_2 v_2$ leaves the tube at the other end at pressure p_2. The corresponding work is $dW_2 = -p_2 dV$, such that the total work done by the pressure is

$$dW = dW_1 + dW_2 = (p_1 - p_2) dV. \tag{10.2.2}$$

This work must correspond to the change in energy of the volume dV of liquid as it moves from position 1 to 2. If the entering end 1 is at a height z_1 the sum of kinetic and potential energy is

$$E_1 = \frac{1}{2}\rho dV v_1^2 + \rho dV g z_1. \tag{10.2.3}$$

When it leaves the tube at position 2 at a height z_2, the energy has changed to

$$E_2 = \frac{1}{2}\rho dV v_2^2 + \rho dV g z_2. \tag{10.2.4}$$

Energy conservation requires

$$dW = E_2 - E_1 \;\Rightarrow\; (p_1 - p_2)dV = \frac{1}{2}\rho dV(v_2^2 - v_1^2) + \rho dV g(z_2 - z_1), \tag{10.2.5}$$

which implies Bernoulli's equation

$$p_1 + \frac{1}{2}\rho v_1^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^2 + \rho g z_2. \tag{10.2.6}$$

This equation can be used to calculate the pressure of a fluid in steady motion. For a fluid at rest ($v_1 = v_2 = 0$) Bernoulli's equation reduces to Pascal's law

$$p_1 = p_2 - \rho g(z_1 - z_2). \tag{10.2.7}$$

10.3 Surface Tension

The molecules in a liquid prefer to be next to other molecules at a distance r_0. This is not always possible for molecules at the surface of the liquid. Their potential energy is hence larger than that of typical molecules in the bulk of the liquid. Consequently, the surface of a liquid costs energy U in proportion to the area A. The constant of proportionality is the so-called surface tension

$$\sigma = \frac{U}{A}. \tag{10.3.1}$$

Appendix A

Physical Units

In physics we encounter a variety of units. Physical units are to a large extent a convention influenced by the historical development of science. In order to understand the physics literature, we must familiarize ourselves with various choices of units, even if the conventions may seem unnatural from today's point of view. Interestingly, there are also natural units which express physical quantities in terms of fundamental constants of Nature: Newton's gravitational constant G, the velocity of light c, and Planck's quantum h. In this appendix, we consider the issue of physical units from a general point of view.

A.1 Units of Time

Time is measured by counting periodic phenomena. The most common periodic phenomenon in our everyday life is the day and, related to that, the year. Hence, it is no surprise that the first precise clock used by humans was the solar system. In our life span, if we stay healthy, we circle around the sun about 80 times, every circle defining one year. During one year, we turn around the earth's axis 365 times, every turn defining one day. When we build a pendulum clock, it helps us to divide the day into $24 \times 60 \times 60 = 86400$ seconds. The second (about the duration of one heart beat) is perhaps the shortest time interval that people care about in their everyday life. However, as physicists we do not stop there, because we need to be able to measure much shorter time intervals, in particular, as we investigate physics of fundamental objects such as atoms or individual elementary particles. Instead of using the solar system as a gigantic mechanical clock, the most accurate modern clock is a tiny quantum mechanical analog of the solar system — an individual cesium atom. Instead of defining 1 sec as one turn around earth's axis divided into 86400 parts, the modern definition of 1 sec corresponds to 9192631770 periods of a particular microwave transition of the cesium atom. The atomic cesium clock is very accurate and defines a reproducible standard of time. Unlike the solar system, this standard could be established anywhere in the Universe. Cesium atoms are fundamental objects which work in the same way everywhere at all times.

A.2 Units of Length

The lengths we care about most in our everyday life are of the order of the size of our body. It is therefore not surprising that, in order to define a standard, some stick — defined to be 1 meter — was deposited near Paris a long time ago. Obviously, this is a completely arbitrary man-made unit. A physicist elsewhere in the Universe would not want to subscribe to that convention. A trip to Paris just to measure a length would be too inconvenient. How can we define a natural standard of length that would be easy to establish anywhere at all times? For example, one could say that the size of our cesium atom sets such a standard. Still, this is not how this is handled. Einstein has taught us that the velocity of light c in vacuum is an absolute constant of Nature, independent of any observer. Instead of referring to the stick in Paris, one now defines the meter through c and the second as

$$c = 2.99792456 \times 10^8 \text{m sec}^{-1} \ \Rightarrow \ 1\text{m} = 3.333564097 \times 10^{-7} c \text{ sec}. \tag{A.2.1}$$

In other words, the measurement of a distance is reduced to the measurement of the time it takes a light signal to pass that distance. Since relativity theory tells us that light travels with the same speed everywhere at all times, we have thus established a standard of length that could easily be used by physicists anywhere in the Universe.

A.3 Units of Mass

Together with the meter stick, a certain amount of platinum-iridium alloy was deposited near Paris a long time ago. The corresponding mass was defined to be one kilogram. Obviously, this definition is as arbitrary as that of the meter. Since the original kilogram has been moved around too often over the past 100 years or so, it has lost some weight and no longer satisfies modern requirements for a standard of mass. One might think that it would be best to declare, for example, the mass of a single cesium atom as an easily reproducible standard of mass. While this is true in principle, it is inconvenient in practical experimental situations. Accurately measuring the mass of a single atom is highly non-trivial. Instead, it was decided to produce a more stable kilogram that will remain constant for the next 100 years or so. Maintaining a standard is important business for experimentalists, but a theorist doesn't worry too much about the arbitrarily chosen amount of matter deposited near Paris.

A.4 Natural Planck Units

Irrespective of practical considerations, it is interesting to think about standards that are natural from a theoretical point of view. There are three fundamental constants of Nature that can help us in this respect. First, to establish a standard of length, we have already used the velocity of light c which plays a central role in the theory of relativity. Quantum mechanics provides us with another fundamental constant — Planck's quantum (divided by 2π)

$$\hbar = \frac{h}{2\pi} = 1.0546 \times 10^{-34} \text{kg m}^2 \text{sec}^{-1}. \tag{A.4.1}$$

As theorists, we are not terribly excited about knowing the value of $\hbar$ in units of kilograms, meters, and seconds, because these are arbitrarily chosen man-made units. Instead, it would be natural to use $\hbar$ itself as a basic unit of dimension energy times time. A third dimensionful fundamental constant that suggests itself through general relativity is Newton's gravitational constant

$$G = 6.6720 \times 10^{-11} \text{kg}^{-1} \text{m}^3 \text{sec}^{-2}. \tag{A.4.2}$$

Using c, $\hbar$, and G we can define natural units also known as Planck units. First there are the Planck time

$$t_{\text{Planck}} = \sqrt{\frac{G\hbar}{c^5}} = 5.3904 \times 10^{-44} \text{sec}, \tag{A.4.3}$$

and the Planck length

$$l_{\text{Planck}} = \sqrt{\frac{G\hbar}{c^3}} = 1.6160 \times 10^{-35} \text{m}, \tag{A.4.4}$$

which represent the shortest times and distances relevant in physics. Today we are very far from exploring such short length- and time-scales experimentally. It is even believed that our classical concepts of space and time will break down at the Planck scale. One may speculate that at the Planck scale space and time become discrete, and that l_{Planck} and t_{Planck} may represent the shortest elementary quantized units of space and time. We can also define the Planck mass

$$M_{\text{Planck}} = \sqrt{\frac{\hbar c}{G}} = 2.1768 \times 10^{-8}\text{kg}, \tag{A.4.5}$$

which is the highest mass scale relevant to elementary particle physics.

Planck units would not be very practical in our everyday life. For example, a lecture would last about $10^{48} t_{\text{Planck}}$, the distance from the lecture hall to the cafeteria would be about $10^{37} l_{\text{Planck}}$, and our lunch would weigh about $10^7 M_{\text{Planck}}$. Still, these are the natural units that Nature suggests to us and it is interesting to ask why we exist at scales so far removed from the Planck scale. For example, we may ask why the Planck mass corresponds to about 10^{-8} kg. In some sense this is just a historical question. The amount of matter deposited near Paris to define the kilogram obviously was an arbitrary man-made unit. However, if we assume that the kilogram was chosen because it is a reasonable fraction of our own weight, it may be phrased as a biological question: Why do intelligent creatures weigh about $10^{10} M_{\text{Planck}}$? We can turn the question into a physics problem when we think about the physical origin of mass. Indeed (up to tiny corrections) the mass of the matter that surrounds us is contained in atomic nuclei which consist of protons and neutrons with masses

$$\begin{aligned} M_p &= 1.67266 \times 10^{-27}\text{kg} = 7.6840 \times 10^{-20} M_{\text{Planck}}, \\ M_n &= 1.67496 \times 10^{-27}\text{kg} = 7.6946 \times 10^{-20} M_{\text{Planck}}. \end{aligned} \tag{A.4.6}$$

Why are protons and neutrons so light compared to the Planck mass? This physics question has actually been understood at least qualitatively using the property of asymptotic freedom of quantum chromodynamics (QCD) — the quantum field theory of quarks and gluons whose interaction energy explains the masses of protons and neutrons. The Nobel prize of the year 2004 was awarded to David Gross, David Politzer, and Frank Wilczek for the understanding of asymptotic freedom. As discussed in chapter III, eq.(A.4.6) also explains why gravity is an extremely weak force.

The strength of electromagnetic interactions is determined by the quantized charge unit e (the electric charge of a proton). In natural Planck units it gives rise to the experimentally determined fine-structure constant

$$\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.036}. \tag{A.4.7}$$

The strength of electromagnetism is determined by this pure number which is completely independent of any man-made conventions. It is a very interesting physics question to ask why α has this particular value. At the moment, physicists have no clue how to answer this question. These days it is popular to refer to the anthropic principle. If α would be different, all of atomic physics and thus all of chemistry would work differently, and life as we know it might be impossible. According to the anthropic principle, we can only live in a part of a Multiverse (namely in our Universe) with a "life-friendly" value of α. The author does not subscribe to this way of thinking. Since we will always be confined to our Universe, we cannot falsify the anthropic argument. In this sense, it does not belong to rigorous scientific thinking. This does not mean that one should not think that way, but it is everybody's private business. The author prefers to remain optimistic and hopes that some day some smart physicist will understand why α takes the above experimentally observed value.

A.5 Units of Charge

As we have seen, in Planck units the strength of electromagnetism is given by the fine-structure constant which is a dimensionless number independent of any man-made conventions. Obviously, the natural unit of charge that Nature suggests to us is the elementary charge quantum e — the electric charge of a single proton. In experiments on macroscopic scales one usually deals with an enormous number of elementary charges at the same time. Just like using Planck units in everyday life is not very practical, measuring charge in units of

$$e = \sqrt{\frac{\hbar c}{137.036}} = 1.5189 \times 10^{-14} \text{kg}^{1/2} \text{m}^{3/2} \text{sec}^{-1} \tag{A.5.1}$$

can also be inconvenient. For this purpose large amounts of charge have also been used to define charge units. For example, one electrostatic unit is defined as

$$1\text{esu} = 2.0819 \times 10^{9} e = 3.1622 \times 10^{-5} \text{kg}^{1/2} \text{m}^{3/2} \text{sec}^{-1}. \tag{A.5.2}$$

This charge definition has the curious property that the Coulomb force between two electrostatic charge units at a distance of 1 centimeter is

$$1 \frac{\text{esu}^2}{\text{cm}^2} = 10^{-5} \text{kg m sec}^{-2} = 10^{-5} \text{N} = 1 \text{dyn}. \tag{A.5.3}$$

Again, the origin of this charge unit is at best of historical interest and just represents another man-made convention.

In this course, we use the so-called Gaussian system in which charge is of dimension mass$^{1/2}$ × length$^{3/2}$ × time^{-1}. This is the system that is most natural from a theorist's point of view. An alternative used in many technical applications and described in several textbooks is the MKS system. Although it is natural and completely sufficient to assign the above dimension to charge, in the MKS system charge is measured in a new independent unit called a Coulomb in honor of Charles-Augustin de Coulomb (1736 - 1806). In MKS units Coulomb's law for the force between two charges q_1 and q_2 at a distance r takes the form

$$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}. \tag{A.5.4}$$

The only purpose of the quantity ε_0 is to compensate the dimension of the charges. For example, in MKS units the elementary charge quantum corresponds to e' with

$$e = \frac{e'}{\sqrt{4\pi\varepsilon_0}}. \tag{A.5.5}$$

The Coulomb is then defined as

$$1\text{Cb} = 6.2414 \times 10^{18} e'. \tag{A.5.6}$$

Comparing with eq.(A.5.2) one obtains

$$1\text{Cb} = 10c\sqrt{4\pi\varepsilon_0}\ \text{sec m}^{-1}\text{esu}. \tag{A.5.7}$$

Appendix B

Vector Algebra

Before we can study mechanics we need to equip ourselves with some necessary mathematical tools such as vector algebra.

B.1 Addition of Vectors and Multiplication by Numbers

A general 3-dimensional vector can be written as

$$\vec{A} = (A_x, A_y, A_z). \tag{B.1.1}$$

Here A_x, A_y, and A_z are the x-, y-, and z-component of the vector $\vec{A}$. The sum of two vectors is again a vector which arises from component-wise addition

$$\vec{A} + \vec{B} = (A_x, A_y, A_z) + (B_x, B_y, B_z) = (A_x + B_x, A_y + B_y, A_z + B_z). \tag{B.1.2}$$

Similarly, a vector can be multiplied by a number $\lambda \in \mathbb{R}$ by multiplying the individual components

$$\lambda\vec{A} = \lambda(A_x, A_y, A_z) = (\lambda A_x, \lambda A_y, \lambda A_z). \tag{B.1.3}$$

The result is again a vector.

It is convenient to introduce three unit-vectors pointing along the coordinate axes

$$\vec{e}_x = (1, 0, 0),\ \ \vec{e}_y = (0, 1, 0),\ \ \vec{e}_z = (0, 0, 1). \tag{B.1.4}$$

Then every vector can be expressed as a linear combination of unit-vectors

$$\vec{A} = (A_x, A_y, A_z) = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z. \tag{B.1.5}$$

B.2 Scalar Product

The scalar product of two vectors

$$\vec{A} = (A_x, A_y, A_z),\ \vec{B} = (B_x, B_y, B_z) \tag{B.2.1}$$

is denoted by $\vec{A} \cdot \vec{B}$ and is defined as

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z. \tag{B.2.2}$$

This is a single number — a scalar — not a vector. The scalar product of a vector with itself is

$$\vec{A} \cdot \vec{A} = A_x^2 + A_y^2 + A_z^2, \tag{B.2.3}$$

which is just the magnitude squared. Geometrically, the scalar product is related to the angle θ between the two vectors by

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\theta. \tag{B.2.4}$$

B.3 Vector Cross Product

The vector cross product of two vectors is another vector defined as

$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y, A_z B_x - A_x B_z, A_x B_y - A_y B_x). \tag{B.3.1}$$

The vector cross product is anti-commutative, i.e.

$$\vec{B} \times \vec{A} = -\vec{A} \times \vec{B}. \tag{B.3.2}$$

Hence, we must pay attention to the order of the two vectors. The vector $\vec{A} \times \vec{B}$ is perpendicular to both $\vec{A}$ and $\vec{B}$. This follows from

$$\begin{aligned} \vec{A} \cdot (\vec{A} \times \vec{B}) &= A_x A_y B_z - A_x A_z B_y + A_y A_z B_x \\ &- A_y A_x B_z + A_z A_x B_y - A_z A_y B_x = 0. \end{aligned} \tag{B.3.3}$$

What is the length of the vector $\vec{A} \times \vec{B}$? To answer this question, we consider

$$\begin{aligned} |\vec{A} \times \vec{B}|^2 &= A_y^2 B_z^2 + A_z^2 B_y^2 - 2A_y B_y A_z B_z \\ &+ A_z^2 B_x^2 + A_x^2 B_z^2 - 2A_z B_z A_x B_x \\ &+ A_x^2 B_y^2 + A_y^2 B_x^2 - 2A_x B_x A_y B_y, \end{aligned} \tag{B.3.4}$$

as well as

$$\begin{aligned} |\vec{A}|^2 |\vec{B}|^2 &= (A_x^2 + A_y^2 + A_z^2)(B_x^2 + B_y^2 + B_z^2), \\ (\vec{A} \cdot \vec{B})^2 &= (A_x B_x + A_y B_y + A_z B_z)^2. \end{aligned} \tag{B.3.5}$$

Combining the above equations, it is easy to convince oneself that

$$|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 - (\vec{A} \cdot \vec{B})^2. \tag{B.3.6}$$

Next we use

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}| \cos\varphi, \tag{B.3.7}$$

where φ is the angle between the vectors $\vec{A}$ and $\vec{B}$, and we find

$$|\vec{A} \times \vec{B}|^2 = |\vec{A}|^2 |\vec{B}|^2 - |\vec{A}|^2 |\vec{B}|^2 \cos^2\varphi = |\vec{A}|^2 |\vec{B}|^2 \sin^2\varphi. \tag{B.3.8}$$

Hence we obtain

$$|\vec{A} \times \vec{B}| = |\vec{A}||\vec{B}| \sin\varphi. \tag{B.3.9}$$

Appendix C

Differentiation and Integration

In this appendix we briefly introduce differentiation and integration as a limit of quotients of finite differences and sums of products. Some rules of differentiation and integration are also discussed.

C.1 Differentiation

The derivative $df(x)/dx$ measures the rate of change of some function $f(x)$ with x and is defined as the limit

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}. \tag{C.1.1}$$

It corresponds to the slope of the curve $f(x)$ at the point x. According to this definition one immediately obtains

$$\frac{d(f+g)(x)}{dx} = \frac{df(x)}{dx} + \frac{dg(x)}{dx}, \tag{C.1.2}$$

as well as

$$\frac{d(\lambda f)(x)}{dx} = \lambda \frac{df(x)}{dx}. \tag{C.1.3}$$

The derivative of a product results as

$$\begin{aligned}\frac{d(fg)(x)}{dx} &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{[f(x+\Delta x)-f(x)]g(x+\Delta)+f(x)[g(x+\Delta x)-g(x)]}{\Delta x}\\ &= \frac{df(x)}{dx}g(x)+f(x)\frac{dg(x)}{dx},\end{aligned} \tag{C.1.4}$$

and the chain rule for nested differentiation takes the form

$$\frac{df(y(x))}{dx}=\frac{df(y)}{dy}\frac{dy(x)}{dx}. \tag{C.1.5}$$

C.2 Integration

The area under the curve $f(x)$ in the interval $[a, b]$ is given by

$$\int_a^b dx\ f(x)=\lim_{\Delta x\to 0}\sum_{x=a}^{b-\Delta x}\Delta x\ f(x). \tag{C.2.1}$$

Integration is the inverse of differentiation. This follows from

$$\begin{aligned}\int_a^b dx\ \frac{df(x)}{dx} &= \lim_{\Delta x\to 0}\sum_{x=a}^{b-\Delta x}\Delta x\ \frac{df(x)}{dx}\\ &= \lim_{\Delta x\to 0}\sum_{x=a}^{b-\Delta x}\Delta x\ \frac{f(x+\Delta x)-f(x)}{\Delta x}\\ &= f(b)-f(b-\Delta x)+f(b-\Delta x)-f(b-2\Delta x)+\ldots\\ &+ f(a+\Delta x)-f(a)=f(b)-f(a).\end{aligned} \tag{C.2.2}$$

Hence, the area under the curve $df(x)/dx$ in the interval $[a, b]$ is given by $f(b) - f(a)$. The formula for partial integration

$$\int_a^b dx\ f(x)\frac{dg(x)}{dx}=f(b)g(b)-f(a)g(a)-\int_a^b dx\ \frac{df(x)}{dx}g(x), \tag{C.2.3}$$

follows from the product rule eq.(C.1.4) combined with eq.(C.2.2), such that indeed

$$\begin{aligned}f(b)g(b)-f(a)g(a) &= \int_a^b dx\ \frac{d(fg)(x)}{dx}\\ &= \int_a^b dx\ \left(\frac{df(x)}{dx}g(x)+f(x)\frac{dg(x)}{dx}\right).\end{aligned} \tag{C.2.4}$$

Appendix D

Some Relevant Functions

In this section we introduce some relevant functions — polynomials, exponential functions, harmonic functions, as well as logarithms — and we discuss some of their basic properties.

D.1 Monomials and Polynomials

Monomials are basic building blocks of more complicated functions. A monomial of degree n is just the n-th power of x

$$f(x) = x^n, \ n \in \mathbb{N}. \tag{D.1.1}$$

The derivative of this monomial is given by

$$\frac{df(x)}{dx} = nx^{n-1}. \tag{D.1.2}$$

This follows directly from the definition of differentiation

$$\begin{aligned} \frac{df(x)}{dx} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x)^n - x^n}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^n + nx^{n-1}\Delta x - x^n}{\Delta x} = nx^{n-1}. \end{aligned} \tag{D.1.3}$$

Polynomials of degree N are linear combinations of monomials of different degrees up to N

$$f(x) = \sum_{n=0}^{N} a_n x^n, \ a_n \in \mathbb{R}. \tag{D.1.4}$$

The derivative of this polynomial is given by

$$\frac{df(x)}{dx} = \sum_{n=0}^{N} a_n n x^{n-1}. \tag{D.1.5}$$

Simple examples of polynomials are the constant function

$$f(x) = a_0, \tag{D.1.6}$$

(which is actually also a monomial), the linear function

$$f(x) = a_0 + a_1 x, \tag{D.1.7}$$

and the quadratic function (i.e. the parabola)

$$f(x) = a_0 + a_1 x + a_2 x^2. \tag{D.1.8}$$

D.2 Exponential Functions

The exponential function is defined by its power series

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \tag{D.2.1}$$

which can be viewed as the $N \to \infty$ limit of a polynomial with $a_n = 1/n!$. Here $n!$ denotes the factorial, i.e.

$$n! = \prod_{i=1}^{n} i = 1 \times 2 \times ... \times n. \tag{D.2.2}$$

The derivative of the exponential function is given by

$$\frac{d \exp(x)}{dx} = \sum_{n=0}^{\infty} \frac{n x^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \exp(x). \tag{D.2.3}$$

Similarly, by using the chain rule, we obtain

$$\frac{d \exp(ax)}{dx} = a \exp(ax). \tag{D.2.4}$$

Hence, the exponential $f(x) = A\exp(ax)$ satisfies the differential equation

$$\frac{df(x)}{dx} = af(x). \tag{D.2.5}$$

Thus, an exponential is proportional to its own rate of change, which is characteristic of numerous phenomena in Nature. An important property of exponential functions is

$$\exp(a+b) = \exp(a)\exp(b). \tag{D.2.6}$$

D.3 Harmonic Functions

Sine and cosine are periodic functions that describe the motion of a harmonic oscillator. Hence, they are known as harmonic functions, which are defined as

$$\cos(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}, \ \sin(x) = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!}. \tag{D.3.1}$$

This immediately implies

$$\frac{d\sin(x)}{dx} = \sum_{n=0}^{\infty}(-1)^n\frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!} = \cos(x), \tag{D.3.2}$$

as well as

$$\begin{aligned}\frac{d\cos(x)}{dx} &= \sum_{n=0}^{\infty}(-1)^n\frac{2nx^{2n-1}}{(2n)!} = \sum_{n=1}^{\infty}(-1)^n\frac{x^{2n-1}}{(2n-1)!} \\ &= -\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{(2n+1)!} = -\sin(x).\end{aligned} \tag{D.3.3}$$

Using the chain rule, we then find

$$\frac{d\sin(ax)}{dx} = a\cos(ax), \ \frac{d\cos(ax)}{dx} = -a\sin(ax), \tag{D.3.4}$$

and, consequently, one obtains

$$\begin{aligned}\frac{d^2\sin(ax)}{dx^2} &= a\frac{d\cos(ax)}{dx} = -a^2\sin(ax), \\ \frac{d^2\cos(ax)}{dx^2} &= a\frac{d\sin(ax)}{dx} = -a^2\cos(ax).\end{aligned} \tag{D.3.5}$$

The function $f(x) = A\cos(ax) + B\sin(ax)$ thus obeys the differential equation

$$\frac{d^2 f(x)}{dx^2} = -a^2 f(x). \tag{D.3.6}$$

The harmonic functions satisfy the following equations

$$\begin{aligned}\cos(a+b) &= \cos(a)\cos(b) - \sin(a)\sin(b),\\ \sin(a+b) &= \cos(a)\sin(b) + \sin(a)\cos(b).\end{aligned} \tag{D.3.7}$$

D.4 Logarithms

The logarithm is the inverse of the exponential, i.e.

$$\exp(\log(x)) = x. \tag{D.4.1}$$

It should be noted that we will always deal with the natural logarithm with base $e = \exp(1)$ which is sometimes denoted by $\ln(x)$. We denote it by $\log(x)$. Applying the chain rule to the above relation, we obtain

$$\frac{d\exp(\log(x))}{dx} = \exp(\log(x))\frac{d\log(x)}{dx} = x\frac{d\log(x)}{dx} = \frac{dx}{dx} = 1. \tag{D.4.2}$$

Hence, the derivative of the logarithm is given by

$$\frac{d\log(x)}{dx} = \frac{1}{x}. \tag{D.4.3}$$

For $|x| < 1$ the logarithm can be represented by the power series

$$\log(1+x) = -\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n}, \tag{D.4.4}$$

such that indeed

$$\begin{aligned}\frac{d\log(1+x)}{dx} &= -\sum_{n=1}^{\infty}(-1)^n\frac{nx^{n-1}}{n} = -\sum_{n=1}^{\infty}(-1)^n x^{n-1}\\ &= \sum_{n=0}^{\infty}(-1)^n x^n = \frac{1}{1+x}.\end{aligned} \tag{D.4.5}$$

Here we have used the formula for a geometric series

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}. \tag{D.4.6}$$

An important property of the logarithm is

$$\log(ab) = \log(a) + \log(b), \tag{D.4.7}$$

which immediately follows from

$$\exp(\log(a) + \log(b)) = \exp(\log(a)) \exp(\log(b)) = ab = \exp(\log(ab)). \tag{D.4.8}$$