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Relativistic and Non-Relativistic Electron in a Magnetic Field

Ratheban Sellaiah

supervised by
Prof. Dr. Uwe-Jens Wiese

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Contents

Abstract

Quantum mechanics is a very important theory, since it has many applications such as in electronic structure. The simplest way to describe particles is by a non-relativistic quantum mechanical wave equation, in the context of the Schrödinger equation. However, particles may travel close to light speed and therefore relativistic effects cannot be neglected. A combination of Special Relativity postulated by Albert Einstein with Quantum Mechanics led to Relativistic Quantum Mechanics and Quantum Field Theory. The Dirac equation, a four-component spinor equation describes these phenomena. Since the spin is involved and it couples with an external magnetic field, the investigation of this effect is very interesting. Specifically, it leads to the degeneracy of energy levels, so-called Landau levels. The aim of this thesis is to find the degree of the degeneracy and how it relates to possible symmetries.

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1 Introduction

The Dirac equation is a relativistic wave equation explored by Paul Dirac in 1928 [1]. It describes spin $\frac{1}{2}$ particles such as electrons and quarks. It led to the prediction of anti-particles, specifically anti-electrons or positrons and which were indeed discovered in 1932 by Carl David Anderson. The Dirac equation is the first theory that made quantum mechanics and special relativity compatible.
It is one of the foundations of quantum electrodynamics and the Standard Model [2]. It also has its use in fields of condensed matter physics, specifically in so-called topological insulators. These are materials, whose bulk is insulating and their surface is metallic. They do have a strong spin-orbit coupling, which is a consequence of the Dirac equation [3].
Famously, the Dirac equation serves its purpose also in the so-called quantum Hall effect in graphene, which will be discussed in the next section. It will be followed by approaching wave mechanics classically and quantum mechanically, where in Subsection 2.4 a set of lowering and raising operators are introduced and the degeneracies of energy levels are discussed in Subsection 2.5. In Section 3, spin-momentum coupling leads to the Pauli equation. Moving to the main part of this thesis involving the Dirac equation, these phenomena are investigated relativistically in Section 4. In Subsection 4.2, the main objective of this thesis is tackled. This part includes the motion of a relativistic particle in a constant magnetic field and therefore the existence of Landau levels. A symmetry operator, which switches between degenerate states, will be investigated in a relativistic hydrogen atom in Subsection 4.4. The foundings will be applied to the case of a charged particle in a constant magnetic field. Finally, the thesis is completed by discussing the results of Johnson and Lippmann in Subsection 4.6.

1.1 Quantum Hall Effect

Before we go deeper into the quantum Hall effect, we need to clarify what the classical Hall effect is. Originally discovered by Edwin Hall [4], it describes the phenomena in which a voltage perpendicular to a current can be measured, provided a magnetic field at a right angle to the current itself is applied. In detail, if a magnet is placed near the current then the magnetic field gets distorted. This leads to negatively charged electrons being deflected to one side and positively charged holes to the other side. This generates an electric field and therefore the Hall voltage between the edges of the sample.
The quantum Hall effect as such, as the name already implies, is a quantized version of the Hall effect. This is well known in a two-dimensional electron system existing at low temperature in a strong magnetic field. It is also called the integer quantum Hall effect due to the fact that the Hall resistance is determined by the universal constant $\frac{h}{e^2}$ and an integer number [5]. Its counterpart, the so-called fractional quantum Hall effect, is caused by the Coulomb interaction between electrons [6]. It is remarkable that the integer quantum Hall effect was observed in graphene and at low energy it can be described by the relativistic Dirac equation [7]
.

Graphene is built as a two-dimensional honeycomb lattice consisting of carbon atoms. Its properties are understood within a tight-binding model, which allows electrons to hop between nearest-neighbor sites.
The focus of this thesis is on theoretical aspects of degeneracies of energies related to spin and relativistic effects.

2 Schrödinger Equation

Before we analyse the main problem, which involves the Dirac equation, we need to understand one fundamental aspect in quantum mechanics. What happens if we leave out relativistic effects and spin and only focus on a classical particle in a constant magnetic field? For that it is important to note that we are in an infinite volume. We treat the problem at first classically, then move to semi-classics, and finally end up inspecting the quantum mechanical version. As from here, to simplify calculations, we set $\hbar = c = 1$.

2.1 Classical Analysis

For this simple case we disregard spin and relativistic effects such that the problem is reduced to an electron of mass M and electric charge $-e$ moving in a constant magnetic field $\vec{B} = B\vec{e}_z$. For that we choose the following vector potential

$$A_x(\vec{x}) = 0, \;\; A_y(\vec{x}) = Bx, \;\; A_z(\vec{x}) = 0. \tag{1}$$

Since the particle moves in a magnetic field, it experiences a Lorentz force. This constrains the electron on a circular orbit with constant radius r. The Lorentz force is described by the following equation

$$\vec{F}(t) = -e\vec{v}(t) \times B\vec{e}_z. \tag{2}$$

When an object is moving on a circular orbit it has an angular velocity ω. Combining $v = \omega r$ and $a = \omega^2 r$ with Newton's equation leads to

$$m\omega^2 r = e\omega r B \Rightarrow \omega = \frac{eB}{M}. \tag{3}$$

An important insight is the fact that the angular velocity is independent of the radius r. Moreover the orbits are closed which is related to the conservation of the Runge-Lenz vector creating a hidden accidental symmetry.

We start by inspecting the Lagrange function of the system to find an accidental symmetry

$$L = \frac{M}{2}\vec{v}^2 - e\vec{A}(\vec{x}) \cdot \vec{v} = \frac{M}{2}(\dot{x}^2 - \dot{y}^2) - eBx\dot{y}. \tag{4}$$

Given the Lagrange function, the associated conjugate momenta are given by

$$p_x = \frac{\partial L}{\partial \dot{x}} = M\dot{x}, \;\; p_y = \frac{\partial L}{\partial \dot{y}} = M\dot{y} - eBx. \tag{5}$$

In fact, it is easily visible that the Lagrange function is independent of y and therefore the latter is a cyclic coordinate. This means that p_y itself is conserved due to translation invariance in the y-direction. Interestingly, a translation invariance in the x-direction also exists, even though x is not a cyclic coordinate. For that reason p_x is not conserved. We can find a conserved quantity by using the Noether theorem and obtain

$$P_x = p_x + eBy. \tag{6}$$

From the Lagrange function we obtain the classical Hamilton function

$$H = \vec{p} \cdot \vec{v} - L = \frac{1}{2M}[p_x^2 + (p_y + eBx)^2]. \tag{7}$$

In contrast to quantum mechanics, in classical physics we consider the Poisson bracket instead of the commutator. One can verify that the Hamilton function obeys the following Poisson brackets $\{H, P_x\} = \{H, P_y\} = \{H, L_z\} = 0$. This means automatically that the 3 generators

$$P_x = p_x + eBy, \;\; P_y = p_y, \;\; L_z = x(p_y + \frac{eB}{2}x) - y(p_x + \frac{eB}{2}y), \tag{8}$$

are conserved. Furthermore the commutators between the generators are as follows

$$\{L, P_x\} = P_y, \;\; \{L, P_y\} = -P_x, \;\; \{P_x, P_y\} = eB. \tag{9}$$

From above, we can conclude that P_x and P_y do not commute in a system with a magnetic field. Next, we need to analyse the Runge-Lenz vector (RLV). The RLV is familiar from the Kepler problem in which two bodies interact by a central force that varies as the inverse square of the distance between them. Then the RLV is conserved [8] .

Similarly, in our problem we have a charged particle in a constant magnetic field moving on a closed circle with a fixed center. The fixed center plays the role of a RLV and is given by

$$\begin{aligned} R_x &= x - \frac{\dot{y}}{v}r = x - \frac{1}{M\omega}(p_y + eBx) = -\frac{P_y}{eB}, \\ R_y &= y + \frac{\dot{x}}{v}r = y + \frac{p_x}{M\omega} = \frac{P_x}{eB}. \end{aligned} \tag{10}$$

We see that there is a proportionality relation between the pairs (R_x,R_y) and (-P_y,P_x). This automatically results in (R_x,R_y) and (P_x,P_y) being orthogonal. Consequently the center of the cyclotron orbit is orthogonal to the spatial momentum. From that some interesting relationships can be deduced

$$\{R_x, P_x\} = -\frac{1}{eB}\{P_y, P_x\} = 1, \;\; \{R_y, P_y\} = \frac{1}{eB}\{P_x, P_y\} = 1, \tag{11}$$

and

$$\{R_x, P_y\} = -\frac{1}{eB}\{P_y, P_y\} = 0, \ \ \{R_y, P_x\} = \frac{1}{eB}\{P_x, P_x\} = 0. \tag{12}$$

Especially, one gets the following relationship

$$\{R_x, R_y\} = \frac{1}{eB}. \tag{13}$$

Similar to the translations, the spatial coordinates of the RLV do not have vanishing Poisson brackets. This means that, for the quantum case, a simultaneous arbitrarily precise measurement of the x- and y-components of the center of a cyclotron orbit is not possible.

Now let us inspect the radius r of the cyclotron orbit

$$r^2 = (x - R_x)^2 + (y - R_y)^2 = \frac{1}{M^2\omega^2}(p_y + eBx)^2 + \frac{p_x^2}{M^2\omega^2} = \frac{2H}{M\omega^2}. \tag{14}$$

One immediately sees the proportional relationship between r^2 and the energy and the former is therefore conserved [9]. This fact is going to be very important later on when we discuss the Dirac equation in the particular case involving a constant magnetic field.

2.2 Semi-classical Analysis

Next we treat the problem semi-classically using the Bohr-Sommerfeld quantization. This, in particular, leads to $L = n$ and from there we arrive at the following conclusion

$$L_z = \frac{eB}{2}r^2 = n \Rightarrow r = \sqrt{\frac{2n}{eB}}. \tag{15}$$

Indeed the radii are now quantized. When we further analyse the resulting energy we get

$$E = H = \frac{1}{2}M\omega^2 r^2 = n\omega. \tag{16}$$

This looks very similar to the problem of a harmonic oscillator, except that an additive constant of $\frac{\omega}{2}$ is missing.

2.3 Quantum Mechanical Analysis

Ultimately, we treat the problem entirely quantum mechanically. First the Schrödinger equation takes the form

$$-\frac{1}{2M}[\partial_x^2 + (\partial_y^2 + ieBx)^2]\Psi(\vec{x}) = E\Psi(\vec{x}). \tag{17}$$

Since $[p_y, H] = 0$ due to translational invariance in the y-direction, one can look for energy eigenstates which are also eigenstates of p_y.

This motivates us to make the following ansatz

$$\Psi(\vec{x}) = \psi(x)\exp(ip_y y). \tag{18}$$

As a result of applying this, we obtain

$$[-\frac{\partial_x^2}{2M} + \frac{1}{2}M\omega^2(x + \frac{p_y}{M\omega})^2]\psi(x) = E\psi(x). \tag{19}$$

This is simply a shifted one-dimensional harmonic oscillator and thus the energy spectrum is the same,

$$E = \omega(n + \frac{1}{2}). \tag{20}$$

In fact, it tells us that the energy of the particle does not depend on its momentum p_y leading to continuous infinite degeneracy of each energy level. Famously they are called Landau levels, named after the Soviet physicist Lev Landau [10]. The energy eigenstates take the form

$$\langle\vec{x}|np_y\rangle = \psi_n(x + \frac{p_y}{M\omega})\exp(ip_y y). \tag{21}$$

Here ψ_n stands for the n^{th} eigenstate of a one-dimensional harmonic oscillator.
In a similar way one finds the eigenstates of the momentum $P_x = -i\partial_x + eBy$,

$$\langle\vec{x}|np_x\rangle = \psi_n(y - \frac{p_x}{M\omega})\exp(ip_x x)\exp(-ieBxy). \tag{22}$$

We know that the center of the cyclotron orbit plays the role of a RLV. Therefore it is interesting to inspect the relevance of it with respect to the degeneracy of the energy levels. Indeed the commutation relations are analogous to the classical case $[H, R_x] = [H, R_y] = [H, L] = 0$. Moreover, we can simply think of the RLV as something that also generates translations. This is easily visible when we consider how the RLV and the angular momentum are given in the quantum mechanical case

$$\begin{gathered} R_x = -\frac{P_y}{eB} = \frac{i\partial_y}{eB}, \quad R_y = \frac{P_x}{eB} = y - \frac{i\partial_x}{eB}, \\ L = x(-i\partial_y + \frac{eBx}{2}) - y(-i\partial_x + \frac{eBy}{2}). \end{gathered} \tag{23}$$

Additionally, similar to the classical case, the radius squared of the cyclotron orbit is again conserved

$$r^2 = (x - R_x)^2 + (y - R_y)^2 = (x - \frac{i\partial_y}{eB})^2 - \frac{\partial_x^2}{e^2B^2} = \frac{2H}{M\omega^2}. \tag{24}$$

In fact, it is useful to express the Hamiltonian as follows

$$H = \frac{1}{2} M\omega^2 r^2. \tag{25}$$

Notably, just as the Poisson bracket $\{R_x,R_y\}$ did not vanish, $[R_x,R_y] = \frac{i}{eB}$ implies that the coordinates of the RLV cannot be measured simultaneously, even though the radius itself is associated to a certain energy and therefore is definite in an energy eigenstate.

Let us inspect the commutation relations between the coordinates of the RLV (R_x,R_y) and the angular momentum L_z.

$$[L_z, R_x] = iR_y, \;\; [L_z, R_y] = iR_x. \tag{26}$$

This looks very similar to the relations we know from the hydrogen atom, which are $[L_z, L_x] = iL_y$ and $[L_z, L_y] = -iL_x$.
Therefore it is natural to introduce

$$R_\pm = R_x \pm iR_y \tag{27}$$

which leads to

$$[L, R_\pm] = \pm R_\pm. \tag{28}$$

From that we can identify R_+ and R_- as raising and lowering operators of L_z.

To complete the set of commutation relations we have

$$[R_+, R_-] = \frac{2}{eB}. \tag{29}$$

2.4 Raising and Lowering Operators

In the previous subsection we have seen similarities between the model of a particle in a constant magnetic field and a one-dimensional harmonic oscillator. Thus it makes sense to try to construct raising and lowering operators for the energy.

We approach this by using the representation of the harmonic oscillator Hamiltonian through its raising and lowering operators,

$$H = \omega(a^\dagger a + \frac{1}{2}), \;\; [a^\dagger, a] = 1. \tag{30}$$

Previously, we found a relation between the Hamiltonian of a particle in a magnetic field and the radius squared of the cyclotron orbit which itself is related to the RLV. We now have the following expression

$$H = \frac{1}{2} M\omega^2 r^2 = \frac{1}{2} M\omega^2 [(x - R_x)^2 + (y - R_y)^2]. \tag{31}$$

From there we find

$$a = \sqrt{\frac{M\omega}{2}}[x - R_x - i(y - R_y)], \;\; a^\dagger = \sqrt{\frac{M\omega}{2}}[x - R_x + i(y - R_y)] \tag{32}$$

as the lowering and raising operators. By using the commutation relations between the angular momentum and the mentioned operators, which are

$$[L, a] = -a, \;\; [L, a^\dagger] = a^\dagger, \tag{33}$$

we clearly see that they also raise and lower the angular momentum. Going back to the previous subsection, we identified R_+ as a raising and R_- as a lowering operator as well. By introducing a slight modification, we obtain

$$b = \sqrt{\frac{M\omega}{2}}R_+, \;\; b^\dagger = \sqrt{\frac{M\omega}{2}}R_-. \tag{34}$$

By inspection we get

$$[L, b] = b, \;\; [L, b^\dagger] = -b^\dagger. \tag{35}$$

From there we see that b raises and $b^\dagger$ lowers the angular momentum. Between the raising and lowering operators the following commutation relations exist

$$[a, b] = [a^\dagger, b] = [a, b^\dagger] = [a^\dagger, b^\dagger] = 0, \;\; [b, b^\dagger] = 1. \tag{36}$$

By comparing the model of a two-dimensional harmonic oscillator with a particle in a magnetic field we conclude the following. Both have a set of two commuting raising and lowering operators, but in contrast to the 2-dimensional harmonic oscillator the Hamiltonian of the particle in a magnetic field consists only of $a^\dagger a$ [9] .

2.5 Different Description of the Hamiltonian

It is possible to describe the Hamiltonian by the following expression

$$H = \frac{1}{2}M\omega^2(R_x^2 + R_y^2) + \omega L = \omega(b^\dagger b + \frac{1}{2} + L) = H_0 + \omega L, \tag{37}$$

where

$$H_0 = \omega(b^\dagger b + \frac{1}{2}) \tag{38}$$

is the Hamiltonian of a 1-dimensional harmonic oscillator. Furthermore, the angular momentum is given by

$$L = a^\dagger a - b^\dagger b. \tag{39}$$

It is important that the raising and lowering operators $b^\dagger$ and b commute with the Hamiltonian H,

$$[H, b] = [H_0, b] + \omega[L, b] = 0, \;\; [H, b^\dagger] = [H_0, b^\dagger] + \omega[L, b^\dagger] = 0. \tag{40}$$

2.6 Energy Spectrum and Eigenstates

The previously found relation between b, $b^\dagger$ and the Hamiltonian leads us to an interesting symmetry.
Consider the eigenstate constructed using the raising operators $a^\dagger$ and $b^\dagger$.

$$|nn'\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} \frac{(b^\dagger)^{n'}}{\sqrt{n'!}} |00\rangle. \tag{41}$$

$|00\rangle$ is the ground state which automatically means that it will be annihilated by both a and b.

$$a|00\rangle = b|00\rangle = 0. \tag{42}$$

The state $|nn'\rangle$ is an eigenstate of the Hamiltonian

$$H|nn'\rangle = \omega(n + \frac{1}{2})|nn'\rangle, \tag{43}$$

and also of the angular momentum

$$L_z|nn'\rangle = (n - n')|nn'\rangle = m|nn'\rangle. \tag{44}$$

We need to emphasize that the quantum number $n \in \mathbb{N}_0$ is non-negative whereas the quantum number $m = n - n' \in \mathbb{Z}$ can be any integer. With this in mind, it is evident that the system has infinitely degenerate Landau levels because states with the same n but different n' have the same energy.

In Section 2.3 we found an infinite degeneracy, which is with respect to the continuous momentum p_y. This is a bit different from the one we just found now. The latter has a discrete set of quantum numbers m and therefore the degeneracy is infinitely countable. This difference comes from the fact that we are operating in two different Hilbert spaces. The states relating to p_y are normalized to δ-functions and thus belong to an extended Hilbert space, whereas the countable ones with m are normalizable.

It is astounding that a single particle in a constant magnetic field has an infinitely degenerate ground state. This fact is commonly related to a spontaneous breakdown of symmetry [9].

3 Pauli Equation – a Non-relativistic Spin $\frac{1}{2}$ Extension of the Schrödinger Equation

Until now, we only inspected the Schrödinger equation involving a constant magnetic field. We now take a look at how the equation changes when we include the spin of the charged particle. Specifically, we are going to consider a spin $\frac{1}{2}$ particle, since we are discussing charged fermions.

3.1 Derivation of the Pauli Hamiltonian

The newly added spin leads to the fact that it is coupled to $\vec{B}$ such that the Hamiltonian looks as follows

$$H = \frac{1}{2M}((\vec{p} + e\vec{A})^2 + e\vec{\sigma} \cdot \vec{B}). \tag{45}$$

Since $\vec{A}$ is the one introduced in eq.(1) and $\vec{B} = (0, 0, B)$, the Hamiltonian becomes

$$H = \frac{1}{2M}(p_x^2 + (py + eBx)^2 + eB\sigma_z) \tag{46}$$

We replace eB by $m\omega$ and get

$$H = \frac{1}{2M}(p_x^2 + (py + M\omega x)^2 + M\omega\sigma_z) \tag{47}$$

After performing some transformations it ends up being

$$H = \frac{1}{2M}p_x^2 + \frac{1}{2}M\omega^2(x + \frac{p_y}{M\omega})^2 + \frac{1}{2}\omega\sigma_z. \tag{48}$$

This is relatively similar to eq.(19) with the addition of having a coupling between the spin and the magnetic field. Therefore $M\omega\sigma_z$ or $eB\sigma_z$ comes with it.
We can further express it in a more familiar fashion as

$$H = \frac{1}{2M}p_x^2 + \frac{1}{2}M\omega^2(x + \frac{p_y}{M\omega})^2 + \omega s_z, \tag{49}$$

where $s_z = \pm\frac{1}{2}$ is the spin of the charged fermion.

4 Relativistic Quantum Mechanics

The Pauli equation considers spin-orbit coupling but no further relativistic effects. Since these effects may play an important role, it is necessary to develop a more complete equation. Dirac managed to find an equation which fulfills the conditions imposed by special relativity.
The so-called Dirac equation is a relativistic extension of the Schrödinger equation to describe spin-$\frac{1}{2}$ fermions. It is a ground-breaking discovery due to its prediction of the existence of antimatter, as Dirac himself denoted the negative energy solutions corresponding to positively charged anti-electrons. The existence of that particle, named positron, was later confirmed by Anderson who discovered it. But how did Dirac find his beautiful equation?

4.1 Dirac Equation of a Free Particle

At first we have the Schrödinger equation which looks as follows

$$-\frac{1}{2M}\nabla^2\Psi + V\Psi = i\frac{\partial\Psi}{\partial t}. \tag{50}$$

This, in fact, is just a different representation of the operators $\vec{p}$ and E, which are the following

$$\vec{p} = -i\vec{\nabla}, E = i\frac{\partial}{\partial t}. \tag{51}$$

This automatically brings us to the well-known energy conservation

$$\frac{\vec{p}^2}{2M} + V = E. \tag{52}$$

In relativistic quantum theory we need to consider the relativistic dispersion relation

$$E^2 = p^2 + M^2. \tag{53}$$

By using eq.(51) and (53), we derive the Klein-Gordon equation

$$\frac{\partial^2 \Psi}{\partial t^2} = \nabla^2 \Psi - M^2 \Psi. \tag{54}$$

We express the wavefunction Ψ as follows

$$\Psi = \psi(\vec{r}) e^{-iEt}. \tag{55}$$

The Klein-Gordon equation is a second order in space and time differential equation and describes bosons.

Let us have a look at the continuity equation

$$\frac{\partial \rho}{\partial t} = -\vec{\nabla} \cdot \vec{j}. \tag{56}$$

Here ρ is the charge density and $\vec{j}$ the current density. It is possible to get an equation of this form if we take the Klein-Gordon equation for Ψ and Ψ^* and multiply each with their respective complex conjugate and the imaginary unit i.

$$i\Psi^* \frac{\partial^2 \Psi}{\partial t^2} = i\Psi^* \nabla^2 \Psi - iM^2 \Psi^* \Psi, \tag{57}$$

$$i\Psi \frac{\partial^2 \Psi^*}{\partial t^2} = i\Psi \nabla^2 \Psi^* - iM^2 \Psi \Psi^*. \tag{58}$$

By subtracting eq.(58) from eq.(57) we obtain

$$i(\Psi^* \frac{\partial^2 \Psi}{\partial t^2} - \Psi \frac{\partial^2 \Psi^*}{\partial t^2}) = i(\Psi^* \nabla^2 \Psi - \Psi \nabla^2 \Psi^*). \tag{59}$$

From there we conclude the following equation

$$i\frac{\partial}{\partial t}(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}) = i\vec{\nabla} \cdot (\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^*). \tag{60}$$

This allows us to easily identify the charge density and the current and finally we have

$$\begin{aligned} \rho &= i(\Psi^* \frac{\partial \Psi}{\partial t} - \Psi \frac{\partial \Psi^*}{\partial t}), \\ j &= -i(\Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^*). \end{aligned} \tag{61}$$

If we plug eq.(55) into eq.(61), we see that $\rho = 2E|\psi|^2$. According to the Klein-Gordon equation (53) negative energy solutions are allowed and thus negative charge densities ρ would exist. This is not an issue since there are negative charges.
Dirac was looking for a wave equation that was of first order in time.
For that reason he proposed the following first-order linear differential equation in time and all spatial coordinates

$$i\frac{\partial}{\partial t}\Psi = \vec{\alpha}\cdot(-i\vec{\nabla})\Psi + M\beta\Psi, \tag{62}$$

where $\vec{\alpha} = (\alpha_x,\alpha_y,\alpha_z)$ with $(x, y, z$ being spatial coordinates) and β are matrices, but their size and the elements have to be found. With $H = i\frac{\partial}{\partial t}$ and $\vec{p} = -i\vec{\nabla}$, Dirac created a much more compact form of the free Hamiltonian

$$H = \vec{\alpha}\cdot\vec{p} + \beta M. \tag{63}$$

But how do the matrices α and β look like? For that it is necessary that the Dirac equation is consistent with the relativistic dispersion relation. By squaring eq.(63) we get the following

$$H^2 = (\vec{\alpha}\cdot\vec{p} + \beta M)^2 = \vec{p}^2 + M^2. \tag{64}$$

With $\vec{p}^2 = p_x^2 + p_y^2 + p_z^2$. The Dirac equation obeys the relativistic dispersion relation if the following conditions are fulfilled

$$\alpha_i^2 = I, \qquad \beta^2 = I, \qquad \alpha_i\alpha_j + \alpha_j\alpha_i = 2\delta_{ij}, \qquad \alpha_i\beta + \beta\alpha_i = 0, \tag{65}$$

where i, j are spatial directions. Ordinary numbers always commute and in eq.(65) we see an anticommuting relation, which automatically implies that α_i and β must be matrices.
We go even further and claim that the dimension of the matrices needs to be even. A simple proof explains it [11]. Let's take eq.(65) and change it slightly

$$\alpha_i\beta = -\beta\alpha_i = -I\beta\alpha_i \tag{66}$$

where I is the 4×4 identity matrix

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{67}$$

Now remember a simple theorem

$$\begin{aligned} \det(AB) &= \det(A)\det(B), \\ \det(\alpha_i\beta) = \det(\alpha_i)\det(\beta) &= \det(-I)\det(\beta)\det(\alpha_i). \end{aligned} \tag{68}$$

The important fact that determinants of matrices are scalars and therefore get canceled at two sides of an equation if they are the same, leaves us with $det(-I) = (-1)^N$ with N being the dimension of the matrix. Furthermore the latter needs to fulfill the following condition

$$1 = (-1)^N. \tag{69}$$

From there we automatically see that the dimension of the matrices needs to be even. Dirac made use of that property and was able to find a set of 4 × 4 matrices. A convenient choice is

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \tag{70}$$

with σ_i being the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{71}$$

4.2 Relativistic Motion of a Particle in a Constant Magnetic Field

In our case we investigate a particle in a constant magnetic field. The adapted Dirac equation looks as follows

$$H_B = \vec{\alpha} \cdot (\vec{p} + e\vec{A}) + \beta M, \tag{72}$$

where again $\vec{A} = Bx\vec{e}_y$. Before we try to find a degeneracy operator for the relativistic magnetic case, we first have to define some operators, which are going to provide the foundation for further calculations.
The momentum operators as such are defined as

$$P_x = p_x + eBy, \quad P_y = p_y, \quad P_z = p_z. \tag{73}$$

Furthermore the total angular momentum is defined as

$$\vec{J} = \vec{L} + \vec{S}, \qquad \vec{S} = \frac{1}{2}\vec{\sigma} I = \frac{1}{2}\vec{\Sigma}, \qquad \vec{L} = \vec{r} \times (\vec{p} + e\vec{A}), \qquad \vec{r} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}. \tag{74}$$

Starting from the Hamiltonian in eq.(72), we investigate whether the system is translation and rotation invariant. As such the momentum and the total angular momentum operators need to commute in the respective directions. First let's check the possibility of translation invariance.
In order to prove this, one needs to investigate the commutator between H_B and $\vec{P}$.
We start by inspecting $[H_B, P_y]$ and $[H_B, P_z]$. We know that $[p_i,\, p_j] = 0$ for $i, j \in \{x, y, z\}$

$$[eB\alpha_y A_y, p_y] = [eB\alpha_y A_y, p_z] = 0. \tag{75}$$

The above relation and the fact that all the other commutators are also zero leads to the conclusion that $[H_B,\, P_y] = [H_B,\, P_z] = 0$ and therefore H_B commutes with both P_y and P_z.
Now $[H_B,\, P_x]$ needs to be checked. There the sticking points are $[eB\alpha_y p_y,\, eBy]$ and $[eB\alpha_y A_y,\, p_x]$. The results are the following

$$[eB\alpha_y p_y, eBy] = -ieB\alpha_y, \quad [eB\alpha_y A_y, p_x] = ieB\alpha_y. \tag{76}$$

Since the eqs.(76) cancel each other out, we conclude that $[H_B,\ P_x] = 0$ holds as well. This is indeed compatible with the non-relativistic analogue.
Now we take a look at the total angular momentum. Due to the magnetic field pointing into the z-direction our guess was that the Hamiltonian must commute with the z-component of the total angular momentum. To start with, we first have a look at the following components,

$$\begin{gathered} L_z = x(p_y + \frac{eB}{2}x) - y(p_x + \frac{eB}{2}y), \\ H_B = \vec{\alpha} \cdot (\vec{p} + e\vec{A}) + \beta M. \end{gathered} \tag{77}$$

If we consider the following relations

$$\begin{aligned} [x, p_x] &= i, \qquad & [y, p_y] &= i, \\ [p_x, x^2] &= -2ix, \qquad & [p_y, y^2] &= -2iy, \end{aligned} \tag{78}$$

we can identify the non-vanishing commutators in $[H_B, L_z]$, which are

$$\begin{aligned} [\alpha_x p_x, x p_y] &= -i\alpha_x p_y, \qquad & [\alpha_x p_x, \frac{eB}{2}x^2] &= -ieB\alpha_x x, \\ [\alpha_y eBx, -y p_x] &= -ieB\alpha_y y, \qquad & [\alpha_y p_y, -y p_x] &= i\alpha_y p_x, \\ & & [\alpha_y p_y, -\frac{eB}{2}y^2] &= ieB\alpha_y y. \end{aligned} \tag{79}$$

After combining these relations, we get

$$[H_B, L_z] = ip_x\alpha_y - i(p_y + eBx)\alpha_x. \tag{80}$$

Next, we inspect $[H_B, S_z]$. The Dirac matrices α_i and the spin matrices S_j share the following commutation relation

$$[\alpha_i, S_j] = i\epsilon_{ijk}\alpha_k, \quad i, j, k \in \{x, y, z\}. \tag{81}$$

Applying this to our problem, we get the following non-vanishing commutators

$$\begin{aligned} [\alpha_x p_x, S_z] &= -ip_x\alpha_y, \qquad & [\alpha_y p_y, S_z] &= ip_y\alpha_x, \\ & & [eBx\alpha_y, S_z] &= eBx\alpha_x. \end{aligned} \tag{82}$$

Again, combining these expressions leads us to

$$[H_B, S_z] = -ip_x\alpha_y + i(p_y + eBx)\alpha_x. \tag{83}$$

Interestingly, eq.(83) has exactly the opposite sign as eq.(80). Therefore they cancel each other out and we have proven

$$[H_B, J_z] = 0. \tag{84}$$

We summarize the findings as follows

$$\begin{aligned} [H_B, P_x] &= 0, \\ [H_B, P_y] &= 0, \\ [H_B, P_z] &= 0, \\ [H_B, J_z] &= 0. \end{aligned} \tag{85}$$

Since all components of the momentum vector commute with the Hamiltonian, they are constants of motion and therefore are conserved. This automatically tells us that the system is translation invariant. The conservation of the total angular momentum component reflects the system's rotational symmetry about the z-axis.
Next, we analyze the commutators between the conserved momenta,

$$\begin{aligned} [P_x, P_y] &= ieB, \\ [P_x, P_z] &= 0, \\ [P_y, P_z] &= 0. \end{aligned} \tag{86}$$

Again, as in the non-relativistic case, the operators P_x and P_y cannot be measured with absolute precision simultaneously. Let's see what happens, if we investigate the commutation relation between the total angular momentum in the z-direction, J_z, and the momenta,

$$\begin{aligned} [J_z, P_x] &= iP_y, \\ [J_z, P_y] &= -iP_x, \\ [J_z, P_z] &= 0. \end{aligned} \tag{87}$$

Note that the spin operator S_z does not contribute to the commutator and we immediately see the similarities with the Poisson brackets in eq.(9). Therefore it is interesting how the Runge-Lenz vector is related to the Hamiltonian. First, we define the spatial coordinates, which are exactly the same as in eq.(23).

$$R_x = -\frac{P_y}{eB}, \qquad R_y = \frac{P_x}{eB}. \tag{88}$$

Due to the above relation we conclude, as we already did in the non-relativistic case, that (R_x, R_y) and (P_x, P_y) are orthogonal. The commutation relation between the coordinates of the Runge-Lenz vector and the components of the momenta are as follows

$$\begin{aligned} [R_x, P_x] &= i, \qquad & [R_y, P_y] &= i, \\ [R_x, P_y] &= 0, \qquad & [R_y, P_x] &= 0. \end{aligned} \tag{89}$$

The results are almost the same as in eq.(11) and eq.(12) with the only difference to eq.(11) being the imaginary unit i. This is expected, since we are discussing the quantum mechanical case.
We already found the result of the Poisson bracket between R_x and R_y in eq.(13). The relativistic quantum mechanical case is similar and therefore

$$[R_x, R_y] = \frac{i}{eB}. \tag{90}$$

4.3 Energy Levels in the Relativistic Case

Now we have a look on the Dirac Hamiltonian described in eq.(72) with the exception that we leave out the z-component of the momentum and set $p_z = 0$. We use the same approach as we did in the classical and quantum mechanical cases to see how the Hamiltonian squared and radius squared of the cyclotron orbit are related to each other.
The reduced Dirac Hamiltonian squared can be expressed as follows

$$H_B^2 = (p_x\alpha_x + (p_y + eBx)\alpha_y + \beta M)^2. \tag{91}$$

The momenta in this Hamiltonian are reduced to the x and y directions.
Using the relations in eq.(65), the remaining terms are

$$H_B^2 = p_x^2 + (p_y + eBx)^2 + M^2 + eB\{\alpha_x p_x, \alpha_y x\}. \tag{92}$$

We need to find the anti-commutator of $\{\alpha_x p_x, \alpha_y x\}$, which gives

$$\{\alpha_x p_x, \alpha_y x\} = 2S_z. \tag{93}$$

All together we end up with the following expression

$$H_B^2 = p_x^2 + (p_y + eBx)^2 + M^2 + 2eBS_z. \tag{94}$$

In eq.(24) we found a relation between the Schrödinger Hamiltonian and the radius squared of the cyclotron orbit. We use that to further simplify the expression in eq. (94), thus leading us to

$$H_B^2 = 2MH + M^2 + 2eBS_z. \tag{95}$$

This automatically means that the total energy squared described by the Dirac Hamiltonian in a constant magnetic field can be expressed as follows

$$E_B^2 = 2ME + M^2 + 2eBs_z. \tag{96}$$

Here, the energy E is that of a harmonic oscillator depicted in eq.(20) and $s_z = \pm\frac{1}{2}$. We use this to rewrite the expression in eq.(96) and get

$$E_B^2 = 2eB(n + \frac{1}{2}) + M^2 + 2eBs_z = 2eB(n + s_z + \frac{1}{2}) + M^2. \tag{97}$$

Since the energy of a harmonic oscillator is part of the Dirac energy and we found energy eigenstates of the former to be infinitely degenerate, this might be the case here as well. The only difference to the non-relativistic case is that the spin now influences the energy.

4.4 Accidental Symmetry of the Relativistic Hydrogen Atom

Let us go back to the days when Dirac was busy with finding symmetry operators. Specifically he was engaged with investigating the accidental symmetry of the hydrogen atom. Let's see how the Dirac Hamiltonian of a relativistic hydrogen atom looks like

$$H_H = \vec{\alpha} \cdot \vec{p} + \beta M - \frac{e^2}{r}. \tag{98}$$

He then discovered that there exists a symmetry which was generated by the following operator

$$K = \beta(\vec{\Sigma} \cdot \vec{L} + 1), \qquad [H, K] = 0, \qquad [\vec{J}, K] = 0. \tag{99}$$

Thus the eigenvalues k of the operator K are derived as follows

$$\begin{aligned} \vec{\sigma} \cdot \vec{L} + \mathbb{I} &= (\vec{L} + \vec{S})^2 - \vec{L}^2 - \vec{S}^2 + \mathbb{I} \\ &= \vec{J}^2 - \vec{L}^2 + \frac{\mathbb{I}}{4}, \\ k &= j(j+1) - l(l+1) + \frac{1}{4} \\ &= j(j+1) - (j \pm \frac{1}{2})(j \pm \frac{1}{2} + 1) + \frac{1}{4} \\ &= \pm(j + \frac{1}{2}). \end{aligned} \tag{100}$$

The coupling of $\vec{L}$ and $\vec{S}$ indicates whether they join so that $j = l + \frac{1}{2}$ or $j = l - \frac{1}{2}$ results. The first one implies $k = j + \frac{1}{2}$ and the latter $k = -(j + \frac{1}{2})$.
This automatically leads to the conclusion that the energy eigenstates are defined by the principal quantum number n as wells as by j, j_z and k.

$$H_H|njj_zk\rangle = E_{nk}|njj_zk\rangle, \qquad K|njj_zk\rangle = k|njj_zk\rangle. \tag{101}$$

Finally, the energy eigenvalues are given by

$$E_{nk} = M(1 + \frac{\alpha^2}{(n - |k| + \sqrt{k^2 - \alpha^2})^2})^{-\frac{1}{2}}. \tag{102}$$

For completeness, $\alpha = 0.0072973525643$ is the fine-structure constant.
The interesting role of k is that there lies an accidental symmetry. Suppose we have two states with both having the same quantum number n. One of them has an orbital angular momentum of l and a total angular momentum of $j = l + \frac{1}{2}$ ($k = j + \frac{1}{2}$) and the other one an orbital angular momentum of $l + 1$ and a total angular momentum of $j = l + 1 - \frac{1}{2}$ ($k = -(j + \frac{1}{2})$). Since the energy only depends on $|k|$, both mentioned states have the same energy and are therefore degenerate. It gives rise to a $2(2j+1)$-fold degeneracy. An exception is the state with the maximal orbital angular momentum $l = n - 1$ and the maximal total angular momentum $j = l + \frac{1}{2} = n - \frac{1}{2}$ which has only a $(2j + 1)$-fold degeneracy.

Moreover, there exists a relativistic equivalent of the Runge-Lenz vector, which is the Johnson-Lippmann operator denoted as follows

$$A = iK\gamma_5(\frac{H}{M} - \beta) - \alpha\vec{\Sigma}\cdot\vec{e}_r, \quad [H, A] = 0, \quad [\vec{J}, A] = 0, \quad \gamma_5 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}. \tag{103}$$

As indicated, the operator A commutes with the Hamiltonian so they have a set of common eigenstates. K anti-commutes with A meaning $\{K, A\} = 0$, while A^2 plays the role of a supersymmetric "Hamiltonian"

$$A^2 = K^2[(\frac{H}{M})^2 - 1] + \alpha^2. \tag{104}$$

The eigenvalues of A^2 are given by

$$a^2 = k^2[(\frac{E_{nk}}{M})^2 - 1] + \alpha^2 = \alpha^2 - \frac{\alpha^2 k^2}{(n - |k| + \sqrt{k^2 - \alpha^2})^2 + \alpha^2} \tag{105}$$

The operator A acts on energy eigenstates as

$$A|njj_zk\rangle = a|njj_z - k\rangle. \tag{106}$$

Interestingly, the operator A relates the two accidentally degenerate states with quantum numbers $\pm k$. Assume having an eigenstate with maximal $j = n - \frac{1}{2}$ then the state gets annihilated by the operator A due to $n = j + \frac{1}{2} = k = |k|$, such that

$$a^2 = \alpha^2 - \frac{\alpha^2 k^2}{k^2 - \alpha^2 + \alpha^2} = 0 \implies a = 0. \tag{107}$$

This indicates that (with the exception for the maximal $j = n - \frac{1}{2}$), all the other states are paired and therefore the supersymmetry is not spontaneously broken [12] [13] .

4.5 Symmetry Operator in the Magnetic Field Case

Now, we follow the same procedure to find similar operators for our problem of a relativistic particle in a constant magnetic field. Specifically we need to find an operator like A which relates, for example, the degenerate states with $s_z = -\frac{1}{2}$, $n = 1$ to $s_z = \frac{1}{2}$, $n = 0$ as well as an operator similar to K.
At first, we remember that the total angular momentum in the z-direction is $j_z = l_z + s_z = l_z \pm \frac{1}{2}$. Now we construct J_z^2

$$J_z^2 = L_z^2 + S_z^2 + 2L_zS_z. \tag{108}$$

Using $l_z = j_z \pm \frac{1}{2}$, we rearrange the above equation and get

$$2L_zS_z = j_z^2 - (j_z \pm \frac{1}{2})^2 - \frac{1}{4} = \mp j_z - \frac{1}{2}. \tag{109}$$

We see a similarity to eq.(100), except that an additional constant of $-\frac{1}{2}$ is still there. We move that to the left side of the equation and get

$$2L_zS_z + \frac{1}{2} = \mp j_z. \tag{110}$$

The only thing left is to recognize the relation between eq.(99) and eq.(100) and to use that for our problem. We make the replacement $\Sigma_z = 2S_z$ and obtain

$$K_B = \beta(L_z\Sigma_z + \frac{1}{2}). \tag{111}$$

The K_B operator needs to commute with the Dirac Hamiltonian and the total angular momentum in the z-direction, J_z. Let's check these relations

$$[K_B, J_z] = [\beta(L_z \cdot \Sigma_z + \frac{1}{2}), L_z + \frac{1}{2}\Sigma_z] = 0. \tag{112}$$

The above relation holds, since all the components commute with each other and therefore the whole expression does. The tricky part is the one with the Dirac Hamiltonian $[H_B, K_B]$. We first focus on $[\alpha_i p_i, \frac{1}{2}\beta]$. The only non-commuting part in this bracket is $[\alpha_i, \beta]$

$$\begin{aligned}[\alpha_i, \beta] = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}\begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} - \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix}\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix} = 2\begin{pmatrix} 0 & -\sigma_i \\ \sigma_i & 0 \end{pmatrix}, \\ [\alpha_i p_i, \frac{1}{2}\beta] = 2\begin{pmatrix} 0 & -\sigma_x p_x - \sigma_y p_y \\ \sigma_x p_x + \sigma_y p_y & 0 \end{pmatrix}.\end{aligned} \tag{113}$$

The commutator $[eB\alpha_y x, \frac{1}{2}\beta]$ can be derived from eq.(113) and we get

$$[eB\alpha_y x, \frac{1}{2}\beta] = eB\begin{pmatrix} 0 & -\sigma_y x \\ \sigma_y x & 0 \end{pmatrix}. \tag{114}$$

From these results we find the following relation

$$[H_B, \frac{1}{2}\beta] = \begin{pmatrix} 0 & -\sigma_x p_x - \sigma_y p_y - eB\sigma_y x \\ \sigma_x p_x + \sigma_y p_y + eB\sigma_y x & 0 \end{pmatrix}. \tag{115}$$

Next we analyze $[\alpha_i p_i, \beta L_z\Sigma_z]$,

$$[\alpha_i p_i, \beta L_z\Sigma_z] = \beta L_z[\alpha_i, \Sigma_z]p_i + \alpha_i\beta[p_i, L_z]\Sigma_z + [\alpha_i, \beta]L_z p_i\Sigma_z. \tag{116}$$

We first take a look at $[\alpha_i, \Sigma_z]$. Using eq.(81) and eq.(74) we get

$$[\alpha_x, \Sigma_z] = -2i\begin{pmatrix} 0 & \sigma_y \\ \sigma_y & 0 \end{pmatrix}, \quad [\alpha_y, \Sigma_z] = 2i\begin{pmatrix} 0 & \sigma_x \\ \sigma_x & 0 \end{pmatrix}. \tag{117}$$

Then we have

$$\beta L_z[\alpha_i, \Sigma_z]p_i = -2iL_z\begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix}p_x + 2iL_z\begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}p_y. \tag{118}$$

Next, we inspect $[p_i, L_z]$. By using eq.(77), we see that

$$\begin{aligned}
[p_x, xp_y] &= -ip_y, \\
[p_x, \frac{eB}{2}x^2] &= -ieBx, \\
[p_y, -yp_x] &= ip_x, \\
[p_y, -\frac{eB}{2}y^2] &= ieBy,
\end{aligned} \tag{119}$$

contribute to the expression. The commutator in the last term of eq.(116) was already found in eq.(113). Since $\{\alpha_i, \beta\} = 0$ and $[\alpha_i, \beta]$ is the expression given by eq.(113), we obtain

$$\alpha_{x,y}\beta = \begin{pmatrix} 0 & -\sigma_{x,y} \\ \sigma_{x,y} & 0 \end{pmatrix}. \tag{120}$$

We go further and calculate $\alpha_{x,y}\beta\Sigma_z$

$$\alpha_{x,y}\beta\Sigma_z = \begin{pmatrix} 0 & -\sigma_{x,y} \\ \sigma_{x,y} & 0 \end{pmatrix} \begin{pmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{pmatrix} = \begin{pmatrix} 0 & -\sigma_{x,y}\sigma_z \\ \sigma_{x,y}\sigma_z & 0 \end{pmatrix}. \tag{121}$$

We use $\sigma_x\sigma_z = -i\sigma_y$ and $\sigma_y\sigma_z = i\sigma_x$

$$\alpha_i\beta[p_i, L_z]\Sigma_z = (p_y + eBx) \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} + (p_x + eBy) \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix}. \tag{122}$$

For $[\alpha_i, \beta]L_z p_i \Sigma_z$, we use the fact that if $\{\alpha_i, \beta\} = 0$ holds, then $[\alpha_i, \beta] = 2\alpha_i\beta$ and eq.(121) to get

$$[\alpha_i, \beta]L_z p_i\Sigma_z = -2iL_z \begin{pmatrix} 0 & -\sigma_y \\ \sigma_y & 0 \end{pmatrix} p_x + 2iL_z \begin{pmatrix} 0 & -\sigma_x \\ \sigma_x & 0 \end{pmatrix} p_y. \tag{123}$$

There is one commutator left from the Hamiltonian to calculate, which is

$$\begin{aligned}
[eB\alpha_y x, \beta L_z\Sigma_z] &= eB\beta L_z[\alpha_y, \Sigma_z]x + eB\alpha_y \cdot \beta[x, L_z]\Sigma_z + eB[\alpha_y, \beta]L_z x\Sigma_z \\
&= 2ieB \begin{pmatrix} 0 & L_z\sigma_x x \\ -L_z\sigma_x x & 0 \end{pmatrix} - ieB \begin{pmatrix} 0 & -y\sigma_y\sigma_z \\ y\sigma_y\sigma_z & 0 \end{pmatrix} \\
&\quad +2ieB \begin{pmatrix} 0 & -L_z\sigma_x x \\ L_z\sigma_x x & 0 \end{pmatrix} = eB \begin{pmatrix} 0 & -y\sigma_x \\ y\sigma_x & 0 \end{pmatrix}.
\end{aligned} \tag{124}$$

After having calculated all the necessary commutators we see which ones cancel each other out. We start with $[\alpha_i p_i, \beta L_z\Sigma_z]$. If we add up eqs.(118) and (123), they cancel each other out.
Eqs.(115), (122), and (124) cancel each other out as well.
We conclude that H_B, commutes with K_B

$$[H_B, K_B] = 0. \tag{125}$$

4.6 Johnson-Lippmann Operator

Analogously to the approach followed in the relativistic hydrogen atom, Johnson and Lippmann published a paper in which they did engage themselves with the relativistic motion in a magnetic field. In addition to our 2-dimensional case, they included p_z as well.
The Hamiltonian is defined as follows

$$H = \rho_1 \vec{\sigma} \cdot \vec{\pi} + \rho_3 M. \tag{126}$$

Here, $\vec{\pi} = (\pi_x, \pi_y, \pi_z)$ is the kinetic momentum expressed as

$$\pi_x = p_x + eBy, \quad \pi_y = p_y, \quad \pi_z = p_z. \tag{127}$$

Moreover $\vec{\sigma} = (\Sigma_x, \Sigma_y, \Sigma_z)$ and

$$\rho_1 = \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \quad \rho_3 = \begin{pmatrix} \mathbb{I} & 0 \\ 0 & -\mathbb{I} \end{pmatrix} = \beta \tag{128}$$

complete the set. By inspection, it is easy to verify that this is consistent with eq.(72).
Since $\vec{\sigma}$ commutes with both ρ_1 and ρ_3, it is possible to find an operator, by forming the product of $\vec{\pi}$ and $\vec{\sigma}$, which commutes with H. This operator takes the form

$$I = \vec{\pi} \cdot \vec{\sigma}. \tag{129}$$

Due to the fact that it commutes with H, it is a conserved quantity and therefore an integral of motion. I and H have a set of common eigenfunctions and these represent states in which the spin is parallel or antiparallel to the momentum.
From there on the Hamiltonian can be expressed in terms of I as

$$H = \rho_1 I + \rho_3 M. \tag{130}$$

By defining F as an eigenvalue of I, it is possible to represent the Hamiltonian as follows

$$H = \rho_1 F + \rho_3 M. \tag{131}$$

By introducing the quantities ν_i

$$\begin{aligned} \nu_1 &= \frac{\rho_1 M - \rho_3 F}{\sqrt{F^2 + M^2}}, \\ \nu_2 &= \rho_2, \\ \nu_3 &= \frac{\rho_1 F + \rho_3 M}{\sqrt{F^2 + M^2}}, \end{aligned} \tag{132}$$

the Hamiltonian becomes

$$H = \nu_3 [F^2 + M^2]^{\frac{1}{2}}. \tag{133}$$

Using the fact that $\nu_i^2 = 1$ and ν_i has the same commutation relations as ρ_i, the eigenvalues of ν_3 are ± 1. Therefore H has the energy eigenvalues

$$E = \pm [F^2 + M^2]^{\frac{1}{2}}. \tag{134}$$

Suppose the eigenvalues $\pm F$ of the operator I exist, then the energy E is degenerate in the sign of F. This is analogous to eq.(102). Therefore an operator might exist, which commutes with the Hamiltonian but anticommutes with I. This operator has the ability to switch between degenerate states similar to the operator A in the relativistic hydrogen atom case.
Another thing we noticed from eq.(97) is that the energy levels have a spin degeneracy. Hence, it is useful to express I by

$$I = \sigma_x \pi_x + \sigma_y \pi_y + \sigma_z p, \tag{135}$$

where p is an eigenvalue of π_z. It is known that $\sigma_x \pi_x + \sigma_y \pi_y$ anticommutes with $\sigma_z p$ and therefore the product of both anticommutes with I. This leads to

$$T = i\rho_3 (I - \sigma_z p)\sigma_z = i\rho_3 (I\sigma_z - p) \tag{136}$$

anticommuting with I and commuting with H.
We know from [14] that the energy eigenvalues are given by

$$E = \pm[p^2 + M^2 + 2eB(n + s_z + \frac{1}{2})]^{\frac{1}{2}}, \tag{137}$$

thus implying that from [15]

$$F = \pm[p^2 + 2eB(n + s_z + \frac{1}{2})]^{\frac{1}{2}}. \tag{138}$$

In fact, we can define a state similar to $|njj_zk\rangle$ in Section 4.4. Let's choose $|njj_zF\rangle$.
Since H and I commute, we know that they have a set of common eigenstates, of which one is $|njj_zF\ \rangle$. Applying I on that state, gives us

$$I|njj_zF\rangle = F|njj_zF\rangle. \tag{139}$$

At the same time applying the operator T, which switches between degenerate states, gives the following result

$$T|njj_zF\rangle = t|njj_z - F\rangle. \tag{140}$$

We were able to reproduce the energy eigenvalues for the system which involves a constant magnetic field in the z direction. This is visible if one sets $p = 0$ in eq. (137) and compares with the square root of eq.(97).
Moreover there exists an operator T, which switches between degenerate states similar to the operator A in the hydrogen atom and an operator I, which is analogous to the operator K in eq.(99).

5 Conclusion

The system consisting of a charged particle in a constant magnetic field has an accidental symmetry. Specifically, by performing a classical analysis, we found that P_x, P_y and L_z are 3 generators which are conserved. Since the classical charged particle moves on a closed circular orbit, the fixed center plays the role of a Runge-Lenz vector. By inspecting the radius squared of the cyclotron orbit circle, we explored that it has a proportionality relation with the energy and therefore is conserved.

Moving to the quantum mechanical case, the procedure is similar to the classical one with the addition that the Landau levels with the same quantum number and different m do have the same energy, such that they are infintely degenerate.

Involving spin and relativistic effects leads to the Dirac equation. Having the spin in this system, the total angular momentum J_z is a conserved quantity. It later was found that the energy depends on the spin of the particle. Analogously to the accidental symmetry of the relativistic hydrogen atom, it was possible to find an operator K_B which generates a symmetry. More importantly, since the spin plays a role for the energy levels, it leads to doubly infinitely degenerate energy levels. This was indeed an interesting exploration since in the classical case it was only infinitely degenerate.
Finally, Johnson and Lippmann found an operator I, which is a symmetry operator and another operator T, that allows us to jump between degenerate states.

References

[1] P. A. M. Dirac, *Quantum Theory of Electron.* Proc. R. Soc. Lond. A, 610 (1928).

[2] S. Coleman, *Quantum Field Theory: Lectures of Sidney Coleman.* World Scientific Publishing Co. Pte. Ltd (2019).

[3] D. A. Pesin, A. H. MacDonald, Nature Materials 11, 409 (2012).

[4] E. H. Hall, Amercian Journal of Mathematics 2, 287 (1879).

[5] K. von Klitzing, Phys. Rev. 58, 519 (1986).

[6] H. L. Stormer, D. C. Tsui, A. C. Gossard, Rev. Mod. Phys. 71, 298 (1999).

[7] A. K. Geim, K. S. Novoselov, Nature Materials 6, 183 (2007).

[8] R. C. O'Connell, K. Jagannathan, Amercian Journal of Physics 71, 243 (2003).

[9] M. H. Al-Hashimi, U.-J. Wiese, Annals of Physics 324, 343 (2008).

[10] L. D. Landau, Z. Phys. 64, 31 (1930).

[11] G. D. Mahan, *Quantum Mechanics in a Nutshell.* Princeton University Press (2009).

[12] M. H. Al-Hashimi and U.-J. Wiese, Annals of Physics 362, 621 (2015).

[13] J. P. Dahl and T. Jorgensen, Int. J. Quant. Chem. 53, 161 (1995).

[14] M. H. Johnson, B. A. Lippmann, Phys. Rev. 76, 828 (1949).

[15] M. H. Johnson, B. A. Lippmann, Phys. Rev. 77, 702 (1950).

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Name/Vorname: Sellaiah Ratheban

Matrikelnummer: 13-918-529

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